Lesson 6

THE LIMIT OF A FUNCTION

In this section we shall consider certain cases of the variation of a function when the argument x approaches a certain limit a or infinity.

Definition 1. Let the function $y=f(x)$ be defined in a certain neighbourhood of a point a or at certain points of this neighbourhood. *The function $y=f(x)$ approaches the limit b $(y \longrightarrow b)$ as x approaches a $(x \longrightarrow a)$*, if for every positive number ε, no matter how small, it is possible to indicate a positive number δ such that for all x, different from a and satisfying the inequality*

$$|x-a|<\delta$$

we have the inequality

$$|f(x)-b|<\varepsilon$$

If b is the *limit of the function* $f(x)$ as $x \longrightarrow a$, we write

$$\lim_{x \to a} f(x)=b$$

or $f(x) \longrightarrow b$ as $x \longrightarrow a$.

If $f(x) \longrightarrow b$ as $x \longrightarrow a$, this is illustrated on the graph of the function $y=f(x)$ as follows (Fig. 31). Since from the inequality $|x-a|<\delta$ there follows the inequality $|f(x)-b|<\varepsilon$, this means that for all points x that are not more distant from the point a than δ, the points M of the graph of the function $y=f(x)$ lie within a band of width 2ε bounded by the lines $y=b-\varepsilon$ and $y=b+\varepsilon$.

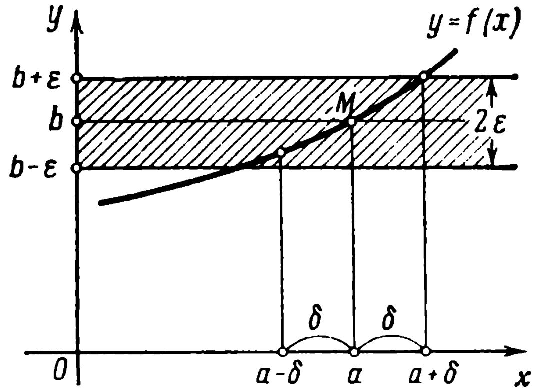

Fig. 31

Note 1. We may also define the limit of the function $f(x)$ as $x \longrightarrow a$ as follows.

Let a variable x assume values such (that is, ordered in such fashion) that if

$$|x^*-a|>|x^{**}-a|$$

then x^{**} is the subsequent value and x^* is the preceding value; but if

$$|\bar{x}^*-a|=|\bar{x}^{**}-a| \text{ and } \bar{x}^*<\bar{x}^{**}$$

then $\bar{x}^{**}$ is the subsequent value and $\bar{x}^*$ is the preceding value. In other words, of two points on a number scale, the subsequent one is that which is closer to the point a; at equal distances, the subsequent one is that which is to the right of the point a.

Let a variable quantity x ordered in this fashion approach the limit a [$x \longrightarrow a$ or $\lim x=a$].

Let us further consider the variable $y=f(x)$. We shall here and henceforward consider that of the two values of a function, the

*Here we mean the values of x that satisfy the inequality $|x-a|<\delta$ and belong to the domain of definition of the function. We will encounter similar circumstances in the future. For instance, when considering the behaviour of a function as $x \longrightarrow \infty$, it may happen that the function is defined only for positive integral values of x. And so in this case $x \longrightarrow \infty$, assuming only positive integral values. We shall not specify this when it comes up later on.

subsequent one is that which corresponds to the subsequent value of the argument.

If, as $x \to a$, a variable y thus defined approaches a certain limit b, we shall write

$$\lim_{x \to a} f(x) = b$$

and we shall say that the function $y = f(x)$ approaches the limit b as $x \to a$.

It is easy to prove that both definitions of the limit of a function are equivalent.

Note 2. If $f(x)$ approaches the limit b_1 as x approaches a certain number a so that x takes on only values less than a we write $\lim\limits_{x \to a-0} f(x) = b_1$ and call b_1 the *limit on the left at the point a of the function*. If x takes on only values greater than a, we write $\lim\limits_{x \to a+0} f(x) = b_2$ and call b_2 the *limit on the right at the point a of the function* (Fig. 32).

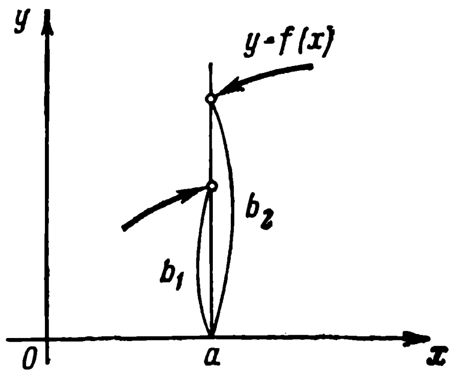

Fig. 32

It can be proved that if the limit on the right and the limit on the left exist and are equal, that is, $b_1 = b_2 = b$, then b will be the limit in the sense of the foregoing definition of a limit at the point a. And conversely, if there exists a limit b of a function at the point a, then there exist limits of the function at the point a both on the right and on the left and they are equal.

Example 1. Let us prove that $\lim\limits_{x \to 2} (3x+1) = 7$. Indeed, let an arbitrary $\varepsilon > 0$ be given; for the inequality $|(3x+1)-7| < \varepsilon$ to be fulfilled it is necessary to have the following inequalities fulfilled:

$$|3x-6| < \varepsilon, \quad |x-2| < \frac{\varepsilon}{3}, \quad -\frac{\varepsilon}{3} < x-2 < \frac{\varepsilon}{3}$$

Thus, given any ε, for all values of x satisfying the inequality $|x-2| < \frac{\varepsilon}{3} = \delta$, the value of the function $3x+1$ will differ from 7 by less than ε. And this means that 7 is the limit of the function as $x \to 2$.

Note 3. For a function to have a limit as $x \to a$, it is not necessary that the function be defined at the point $x = a$. When finding the limit we consider the values of the function in the neighbourhood of the point a that are different from a; this is clearly illustrated in the following case.

Example 2. We shall prove that $\lim\limits_{x \to 2} \frac{x^2-4}{x-2} = 4$. Here, the function $\frac{x^2-4}{x-2}$ is not defined for $x = 2$.

It is necessary to prove that for an arbitrary ε there will be a δ such that the following inequality will be fulfilled:

$$\left|\frac{x^2-4}{x-2}-4\right|<\varepsilon \tag{1}$$

if $|x-2|<\delta$. But when $x\neq 2$ inequality (1) is equivalent to the inequality

$$\left|\frac{(x-2)(x+2)}{x-2}-4\right|=|(x+2)-4|<\varepsilon$$

or

$$|x-2|<\varepsilon \tag{2}$$

Thus, for an arbitrary ε, inequality (1) will be fulfilled if inequality (2) is fulfilled (here, $\delta=\varepsilon$), which means that the given function has the number 4 as its limit as $x\longrightarrow 2$.

Let us now consider certain cases of variation of a function as $x\longrightarrow\infty$.

Definition 2. *The function $f(x)$ approaches the limit b as $x\longrightarrow\infty$* if for each arbitrarily small positive number ε it is possible to indicate a positive number N such that for all values of x that satisfy the inequality $|x|>N$ the inequality $|f(x)-b|<\varepsilon$ will be fulfilled.

Example 3. We will prove that

$$\lim_{x\to\infty}\left(\frac{x+1}{x}\right)=1$$

or

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)=1$$

It is necessary to prove that, for an arbitrary ε, the following inequality is fulfilled

$$\left|\left(1+\frac{1}{x}\right)-1\right|<\varepsilon \tag{3}$$

provided $|x|>N$, where N is determined by the choice of ε. Inequality (3) is equivalent to the following inequality: $\left|\frac{1}{x}\right|<\varepsilon$, which will be fulfilled if

$$|x|>\frac{1}{\varepsilon}=N$$

which means that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)=\lim\limits_{x\to\infty}\frac{x+1}{x}=1$ (Fig. 33).

If we know the meanings of the symbols $x\longrightarrow+\infty$ and $x\longrightarrow-\infty$, the meanings of the following expressions are obvious:

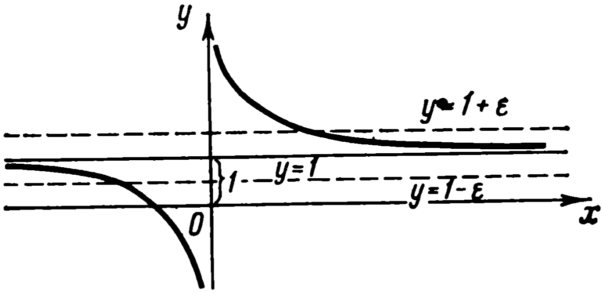

Fig. 33

"$f(x)$ approaches b as $x \to +\infty$" and
"$f(x)$ approaches b as $x \to -\infty$" or, in symbols,

$$\lim_{x \to +\infty} f(x) = b,$$

$$\lim_{x \to -\infty} f(x) = b$$