Lesson 5

LIMIT. CONTINUITY OF A FUNCTION

THE LIMIT OF A VARIABLE. AN INFINITELY LARGE VARIABLE

In this section we shall consider ordered variables that vary in a special way defined as follows: "the variable approaches a limit". Throughout the remainder of the course, the concept of the limit of a variable will play a fundamental role, for it is intimately bound up with the basic concepts of mathematical analysis, such as derivative, integral, etc.

Definition 1. A constant number a is said to be the *limit* of a variable x, if for every preassigned arbitrarily small positive number ε it is possible to indicate a value of the variable x such that all subsequent values of the variable will satisfy the inequality

$$|x-a|<\varepsilon$$

If the number a is the limit of the variable x, one says that x approaches the limit a; in symbols we have

$$x \longrightarrow a \text{ or } \lim x = a$$

In geometric terms, limit may be defined as follows.

The constant number a is the *limit* of the variable x if for any preassigned arbitrarily small neighbourhood with centre in the point a and with radius ε there is a value of x such that all points corresponding to subsequent values of the variable will be within this neighbourhood (Fig. 28).

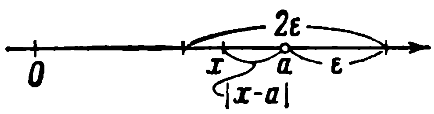

Fig. 28

Let us consider several cases of variables approaching limits.

Example 1. The variable x takes on the successive values

$$x_1=1+1,\ x_2=1+\frac{1}{2},\ x_3=1+\frac{1}{3},\ \ldots,\ x_n=1+\frac{1}{n},\ \ldots$$

We shall prove that this variable has unity as its limit. We have

$$|x_n-1|=\left|\left(1+\frac{1}{n}\right)-1\right|=\frac{1}{n}$$

For any ε, all subsequent values of the variable beginning with n, where $\frac{1}{n} < \varepsilon$, or $n > \frac{1}{\varepsilon}$, will satisfy the inequality $|x_n - 1| < \varepsilon$, and the proof is complete. It will be noted here that the variable quantity decreases as it approaches the limit.

Example 2. The variable x takes on the successive values

$$x_1 = 1 - \frac{1}{2}, \quad x^2 = 1 + \frac{1}{2^2}, \quad x_3 = 1 - \frac{1}{2^3}, \quad x_4 = 1 + \frac{1}{2^4}, \quad \ldots, \quad x_n = 1 + (-1)^n \frac{1}{2^n}, \quad \ldots$$

This variable has a limit of unity. Indeed,

$$|x_n - 1| = \left| \left(1 + (-1)^n \frac{1}{2^n}\right) - 1 \right| = \frac{1}{2^n}$$

For any ε, beginning with n, which satisfies the relation $\frac{1}{2^n} < \varepsilon$, from which it follows that

$$2^n > \frac{1}{\varepsilon}, \quad n \log 2 > \log \frac{1}{\varepsilon} \text{ or } n > \frac{\log \frac{1}{\varepsilon}}{\log 2},$$

all subsequent values of x will satisfy the relation $|x_n - 1| < \varepsilon$. It will be noted here that the values of the variable are greater than or less than the limit, and the variable approaches its limit by "oscillating about it".

Note 1. As was pointed out in Sec. 1.3, a constant quantity c is frequently regarded as a variable whose values are all the same: $x = c$.

Obviously, the limit of a constant is equal to the constant itself, since we always have the inequality $|x - c| = |c - c| = 0 < \varepsilon$ for any ε.

Note 2. From the definition of a limit it follows that a variable cannot have two limits. Indeed, if $\lim x = a$ and $\lim x = b$ $(a < b)$, then x must satisfy, at one and the same time, two inequalities:

$$|x - a| < \varepsilon \text{ and } |x - b| < \varepsilon$$

for an arbitrarily small ε; but this is impossible if $\varepsilon < \frac{b-a}{2}$ (Fig. 29).

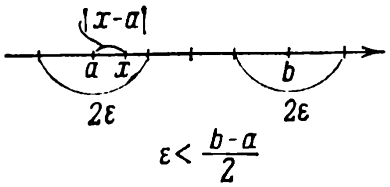

Fig. 29

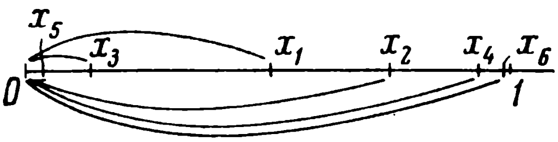

Fig. 30

Note 3. One should not think that every variable has a limit. Let the variable x take on the following successive values (Fig. 30):

$$x_1 = \frac{1}{2}, \quad x_2 = 1 - \frac{1}{4}, \quad x_3 = \frac{1}{8}, \quad \ldots, \quad x_{2k} = 1 - \frac{1}{2^{2k}}, \quad x_{2k+1} = \frac{1}{2^{2k+1}}$$

For k sufficiently large, the value x_{2k} and all subsequent values with even labels will differ from unity by as small a number as we please, while the next value x_{2k+1} and all subsequent values of x with odd labels will differ from zero by as small a number as we please. Consequently, the variable x does not approach a limit.

In the definition of a limit it is stated that if the variable approaches the limit a, then a is a constant. But the word "approaches" is used also to describe another type of variation of a variable, as will be seen from the following definition.

Definition 2. A variable x approaches infinity if for every preassigned positive number M it is possible to indicate a value of x such that, beginning with this value, all subsequent values of the variable satisfy the inequality $|x| > M$.

If the variable x approaches infinity, it is called an *infinitely large* variable and we write $x \longrightarrow \infty$.

Example 3. The variable x takes on the values

$$x_1 = -1,\ x_2 = 2,\ x_3 = -3,\ \ldots,\ x_n = (-1)^n n,\ \ldots$$

This is an infinitely large variable quantity, since for an arbitrary $M > 0$ all values of the variable, beginning with a certain one, are greater than M in absolute value.

The variable x "approaches plus infinity", $x \longrightarrow +\infty$, if for an arbitrary $M > 0$ all subsequent values of the variable, beginning with a certain one, satisfy the inequality $M < x$.

An example of a variable quantity approaching plus infinity is the variable x that takes on the values $x_1 = 1$, $x_2 = 2$, $\ldots$, $x_n = n$, $\ldots$.

A variable approaches minus infinity, $x \longrightarrow -\infty$, if for an arbitrary $M > 0$ all subsequent values of the variable, beginning with a certain one, satisfy the inequality $x < -M$.

For example, a variable x that assumes the values $x_1 = -1$, $x_2 = -2$, $\ldots$, $x_n = -n$, $\ldots$, approaches minus infinity.