1.10. POLAR COORDINATE SYSTEM

The position of a point in a plane may be determined by means of a so-called *polar coordinate system.*

We choose a point O in a plane and call it the *pole;* the half-line issuing from this point is called the *polar axis.* The position of the point M in the plane may be specified by two numbers: the number ρ, which expresses the distance of M from the pole, and the number φ, which is the angle formed by the line segment OM and the polar axis. The positive direction of the angle φ is reckoned counterclockwise. The numbers ρ and φ are called the *polar coordinates* of the point M (Fig. 23).

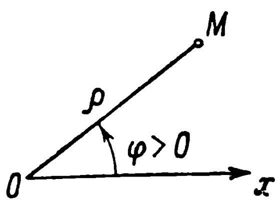

Fig. 23

We will always take the radius vector ρ to be nonnegative. If the polar angle φ is taken within the limits $0 \leqslant \varphi < 2\pi$, then to each point of the plane (with the exception of the pole) there corresponds a definite number pair ρ and φ. For the pole, $\rho = 0$ and φ is arbitrary.

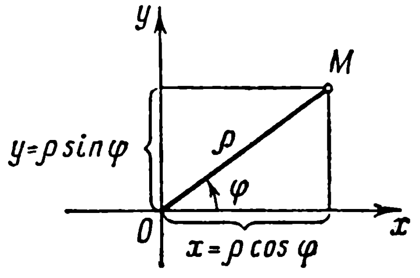

Fig. 24

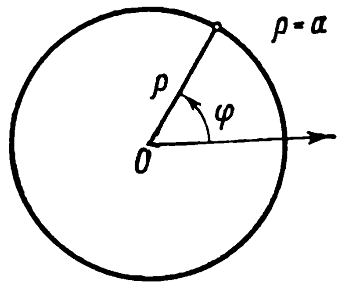

Fig. 25

Let us now see how the polar and rectangular Cartesian coordinates are related. Let the origin of the rectangular coordinate system coincide with the pole, and the positive direction of the x-axis, with the polar axis. From Fig. 24 it follows directly that

$$x = \rho \cos\varphi, \quad y = \rho \sin\varphi$$

and, conversely, that

$$\rho = \sqrt{x^2 + y^2}, \quad \tan\varphi = \frac{y}{x}$$

Note. To find φ, it is necessary to take into account the quadrant in which the point is located and then take the corresponding value of φ. The equation $\rho = F(\varphi)$ in polar coordinates defines a certain line.

Example 1. The equation $\rho = a$, where $a = \text{const}$, defines in polar coordinates a circle with centre in the pole and with radius a. The equation of this circle (Fig. 25) in a rectangular coordinate system situated as shown in Fig. 24 is

$$\sqrt{x^2+y^2} = a \text{ or } x^2+y^2 = a^2$$

Example 2. $\rho = a\varphi$, where $a = \text{const}$.

Let us tabulate the values of ρ for certain values of φ

φ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{3}{2}\pi$	2π	3π	4π
ρ	0	$\approx 0.78a$	$\approx 1.57a$	$\approx 2.36a$	$\approx 3.14a$	$\approx 4.71a$	$\approx 6.28a$	$\approx 9.42a$	$\approx 12.56a$

The corresponding curve is shown in Fig. 26. It is called the *spiral of Archimedes.*

Example 3. $\rho = 2a \cos \varphi$.

This is the equation of a circle of radius a, the centre of which is at the point $\rho_0 = a$, $\varphi = 0$ (Fig. 27). Let us write the equation of this circle in rect-

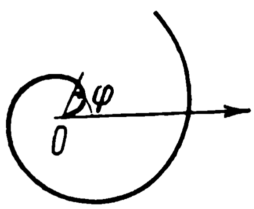

Fig. 26

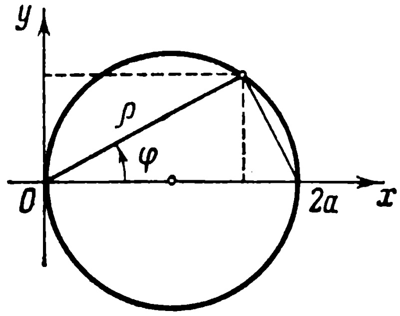

Fig. 27

angular coordinates. Substituting $\rho = \sqrt{x^2+y^2}$, $\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}$ into the given equation, we get

$$\sqrt{x^2+y^2} = 2a\frac{x}{\sqrt{x^2+y^2}}$$

or

$$x^2+y^2-2ax = 0$$

Construct curves given by the polar equations:

41. $\rho = \frac{a}{\varphi}$ (*hyperbolic spiral*). **42.** $\rho = a^{\varphi}$ (*logarithmic spiral*). **43.** $\rho = a\sqrt{\cos 2\varphi}$ (*lemniscate*). **44.** $\rho = a(1-\cos\varphi)$ (*cardioid*). **45.** $\rho = a \sin 3\varphi$.