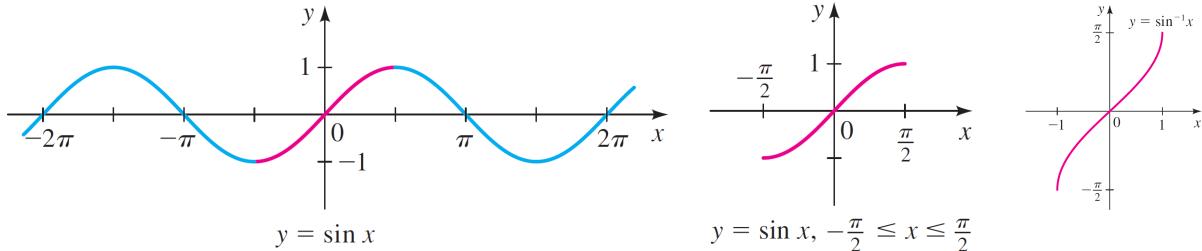


Lesson 3

Inverse Trigonometric Functions and Their Graphs

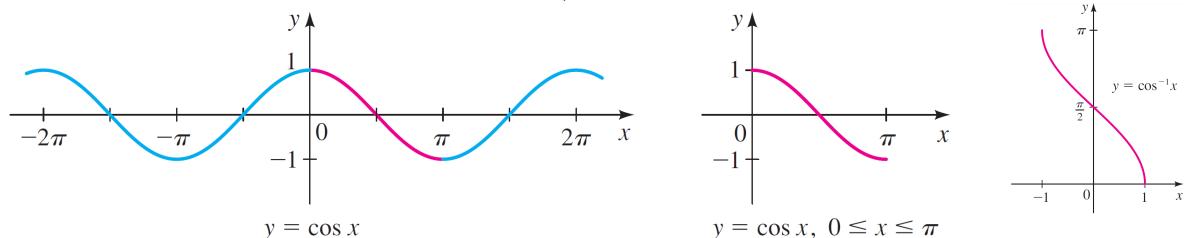
DEFINITION: The **inverse sine function**, denoted by $\sin^{-1} x$ (or $\arcsin x$), is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$



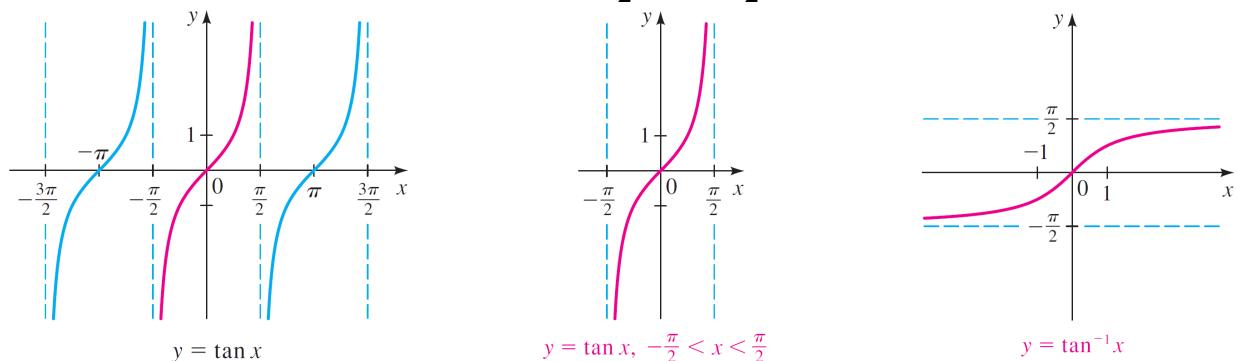
DEFINITION: The **inverse cosine function**, denoted by $\cos^{-1} x$ (or $\arccos x$), is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \leq x \leq \pi$$



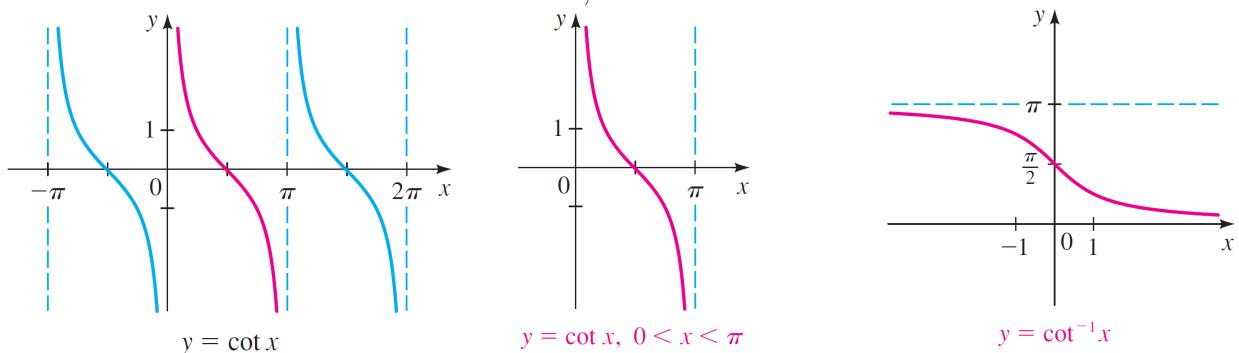
DEFINITION: The **inverse tangent function**, denoted by $\tan^{-1} x$ (or $\arctan x$), is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$



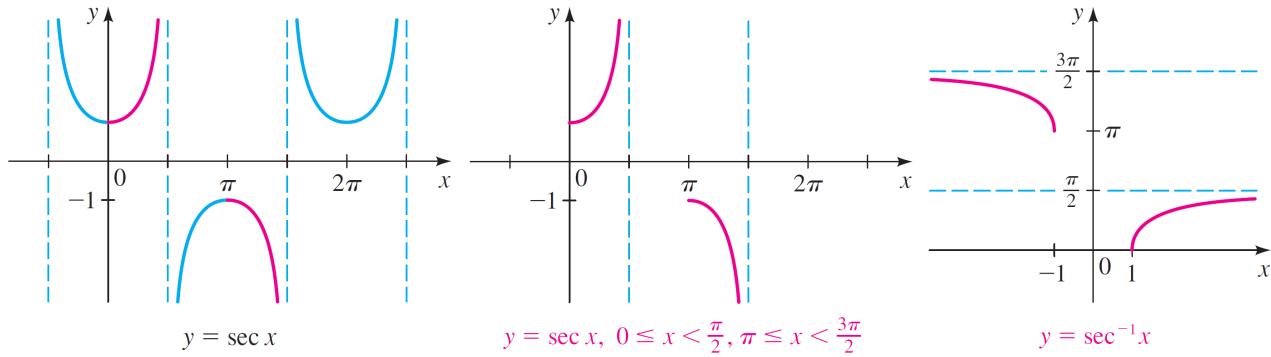
DEFINITION: The **inverse cotangent function**, denoted by $\cot^{-1} x$ (or $\text{arccot } x$), is defined to be the inverse of the restricted cotangent function

$$\cot x, \quad 0 < x < \pi$$



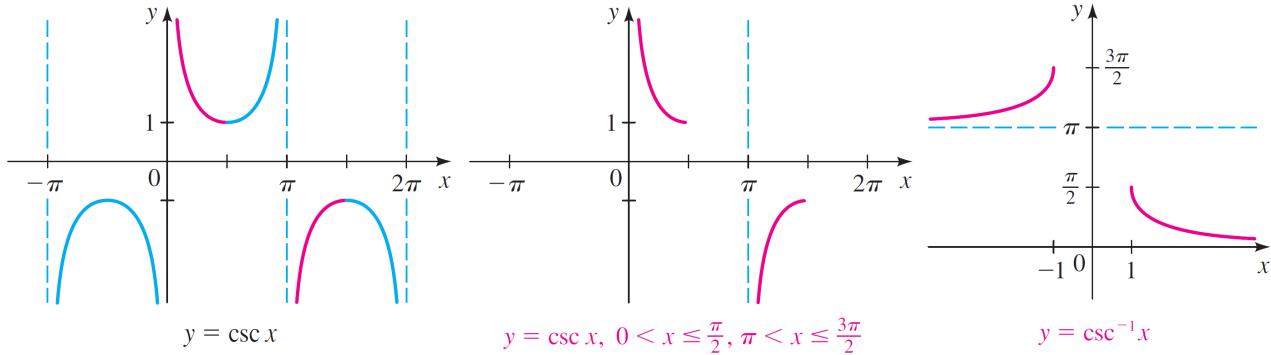
DEFINITION: The **inverse secant function**, denoted by $\sec^{-1} x$ (or $\text{arcsec } x$), is defined to be the inverse of the restricted secant function

$$\sec x, \quad x \in [0, \pi/2) \cup [\pi, 3\pi/2) \quad [\text{or } x \in [0, \pi/2) \cup (\pi/2, \pi] \text{ in some other textbooks}]$$



DEFINITION: The **inverse cosecant function**, denoted by $\csc^{-1} x$ (or $\text{arccsc } x$), is defined to be the inverse of the restricted cosecant function

$$\csc x, \quad x \in (0, \pi/2] \cup (\pi, 3\pi/2] \quad [\text{or } x \in [-\pi/2, 0) \cup (0, \pi/2] \text{ in some other textbooks}]$$



IMPORTANT: Do not confuse

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \cot^{-1} x, \quad \sec^{-1} x, \quad \csc^{-1} x$$

with

$$\frac{1}{\sin x}, \quad \frac{1}{\cos x}, \quad \frac{1}{\tan x}, \quad \frac{1}{\cot x}, \quad \frac{1}{\sec x}, \quad \frac{1}{\csc x}$$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2) \cup (\pi, 3\pi/2)$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

EXAMPLES:

(a) $\sin^{-1} 1 = \frac{\pi}{2}$, since $\sin \frac{\pi}{2} = 1$ and $\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\sin^{-1}(-1) = -\frac{\pi}{2}$, since $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(c) $\sin^{-1} 0 = 0$, since $\sin 0 = 0$ and $0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(d) $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$, since $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(e) $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$, since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

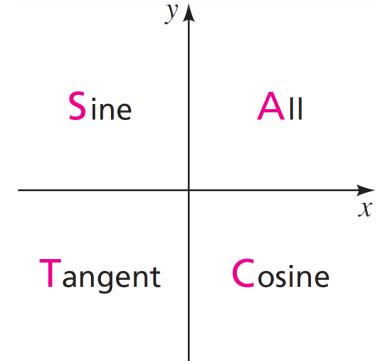
(f) $\sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$, since $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAMPLES:

$$\cos^{-1} 0 = \frac{\pi}{2}, \quad \cos^{-1} 1 = 0, \quad \cos^{-1}(-1) = \pi, \quad \cos^{-1} \frac{1}{2} = \frac{\pi}{3}, \quad \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \tan^{-1}(-1) = -\frac{\pi}{4}, \quad \tan^{-1} \sqrt{3} = \frac{\pi}{3}, \quad \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad \tan^{-1} \left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.



FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.

Solution: We have

$$\sec^{-1} 1 = 0, \quad \sec^{-1}(-1) = \pi, \quad \sec^{-1}(-2) = \frac{4\pi}{3}$$

since

$$\sec 0 = 1, \quad \sec \pi = -1, \quad \sec \frac{4\pi}{3} = -2$$

and

$$0, \pi, \frac{4\pi}{3} \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

Note that $\sec \frac{2\pi}{3}$ is also -2 , but

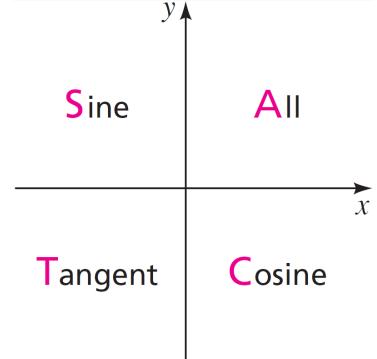
$$\sec^{-1}(-2) \neq \frac{2\pi}{3}$$

since

$$\frac{2\pi}{3} \notin \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

EXAMPLES: Find

$$\tan^{-1} 0 \quad \cot^{-1} 0 \quad \cot^{-1} 1 \quad \sec^{-1} \sqrt{2} \quad \csc^{-1} 2 \quad \csc^{-1} \frac{2}{\sqrt{3}}$$



FUNCTION	DOMAIN	RANGE	t	$\sin t$	$\cos t$	$\tan t$	$\sec t$	$\csc t$	$\cot t$
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	0	0	1	0	—	1	—
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$	$\frac{\pi}{2}$	1	0	—	1	—	0
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$							

EXAMPLES: We have

$$\tan^{-1} 0 = 0, \quad \cot^{-1} 0 = \frac{\pi}{2}, \quad \cot^{-1} 1 = \frac{\pi}{4}, \quad \sec^{-1} \sqrt{2} = \frac{\pi}{4}, \quad \csc^{-1} 2 = \frac{\pi}{6}, \quad \csc^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{3}$$

EXAMPLES: Evaluate

$$(a) \sin \left(\arcsin \frac{\pi}{6} \right), \arcsin \left(\sin \frac{\pi}{6} \right), \text{ and } \arcsin \left(\sin \frac{7\pi}{6} \right).$$

$$(b) \sin \left(\arcsin \frac{\pi}{7} \right), \arcsin \left(\sin \frac{\pi}{7} \right), \text{ and } \arcsin \left(\sin \frac{8\pi}{7} \right).$$

$$(c) \cos \left(\arccos \left(-\frac{2}{5} \right) \right), \arccos \left(\cos \frac{2\pi}{5} \right), \text{ and } \arccos \left(\cos \frac{9\pi}{5} \right).$$

Solution: Since $\arcsin x$ is the inverse of the restricted sine function, we have

$$\sin(\arcsin x) = x \text{ if } x \in [-1, 1] \quad \text{and} \quad \arcsin(\sin x) = x \text{ if } x \in [-\pi/2, \pi/2]$$

Therefore

$$(a) \sin \left(\arcsin \frac{\pi}{6} \right) = \arcsin \left(\sin \frac{\pi}{6} \right) = \frac{\pi}{6}, \text{ but}$$

$$\arcsin \left(\sin \frac{7\pi}{6} \right) = \arcsin \left(-\frac{1}{2} \right) = -\frac{\pi}{6}$$

or

$$\arcsin \left(\sin \frac{7\pi}{6} \right) = \arcsin \left(\sin \left(\pi + \frac{\pi}{6} \right) \right) = \arcsin \left(-\sin \frac{\pi}{6} \right) = -\arcsin \left(\sin \frac{\pi}{6} \right) = -\frac{\pi}{6}$$

$$(b) \sin \left(\arcsin \frac{\pi}{7} \right) = \arcsin \left(\sin \frac{\pi}{7} \right) = \frac{\pi}{7}, \text{ but}$$

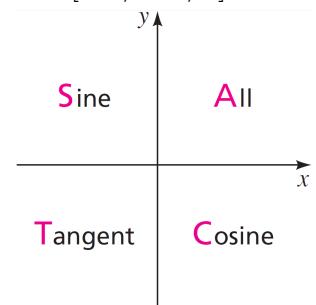
$$\arcsin \left(\sin \frac{8\pi}{7} \right) = \arcsin \left(\sin \left(\frac{\pi}{7} + \pi \right) \right) = \arcsin \left(-\sin \frac{\pi}{7} \right) = -\arcsin \left(\sin \frac{\pi}{7} \right) = -\frac{\pi}{7}$$

(c) Similarly, since $\arccos x$ is the inverse of the restricted cosine function, we have

$$\cos(\arccos x) = x \text{ if } x \in [-1, 1] \quad \text{and} \quad \arccos(\cos x) = x \text{ if } x \in [0, \pi]$$

$$\text{Therefore } \cos \left(\arccos \left(-\frac{2}{5} \right) \right) = -\frac{2}{5} \text{ and } \arccos \left(\cos \frac{2\pi}{5} \right) = \frac{2\pi}{5}, \text{ but}$$

$$\arccos \left(\cos \frac{9\pi}{5} \right) = \arccos \left(\cos \left(2\pi - \frac{\pi}{5} \right) \right) = \arccos \left(\cos \frac{\pi}{5} \right) = \frac{\pi}{5}$$



Calculate

155. $\cos(\arcsin 1)$.

156. $\sin(\arccos 0,8)$:

157. $\sin\left(2\arccos \frac{1}{4}\right)$.

158. $\operatorname{tg}\left(\arcsin \frac{2}{3}\right)$.

159. $\cos\left(\arcsin \frac{3}{5} - \arccos \frac{5}{13}\right)$.

160. $\sin\left(\operatorname{arctg} \frac{1}{3}\right)$.

161. $\sin\left(\arcsin \frac{1}{3} + \arcsin \frac{1}{4}\right)$.

162. $\operatorname{tg}\left(2\operatorname{arc tg}\left(-\frac{1}{2}\right)\right)$.

163. $\arcsin(\sin 11)$.

164. $\arccos(\cos 7)$.

165. $\arcsin(\cos 8)$.

166. $\operatorname{arctg}(\operatorname{tg} 25)$.

167. $\operatorname{arctg}(\operatorname{ctg} 4)$.

168. $\operatorname{arcctg}(\operatorname{ctg} 17)$.

Prove

169. $\arcsin \frac{4}{5} = \arccos \frac{3}{5} = \operatorname{arctg} \frac{4}{3} = \operatorname{arcctg} \frac{3}{4}$.

170. $\arccos\left(-\frac{9}{41}\right) = \pi - \arcsin \frac{40}{41}$.

171. $\arcsin\left(-\frac{7}{25}\right) = -\operatorname{arctg} \frac{7}{24}$.

172. $2\operatorname{arctg} \frac{1}{4} + \operatorname{arctg} \frac{7}{23} = \frac{\pi}{4}$.

173. $\pi - \arcsin 0,9 = 2\operatorname{arctg} 4$.

174. $\operatorname{arctg} \frac{2}{3} + \operatorname{arctg} \frac{1}{5} = \frac{\pi}{4}$.

175. $\cos(2\arccos x) = 2x^2 - 1, |x| \leq 1$.

176. $\sin(3\arcsin x) = 3x - 4x^3, |x| \leq 1$.

177. $\arccos \frac{1-x^2}{1+x^2} = 2|\operatorname{arctg} x|, |x| < \infty$.

178. $\operatorname{ctg}(\operatorname{arctg} x) = \frac{1}{x}, 0 < |x| < \infty$.

$$179. \arctg \frac{1}{x} = \begin{cases} \operatorname{arcctg} x, & x > 0, \\ \operatorname{arcctg} x - \pi, & x < 0. \end{cases}$$

$$180. \arctg \frac{1+x}{1-x} = \begin{cases} \operatorname{arcctg} x + \frac{\pi}{4}, & x < 1, \\ \operatorname{arcctg} x - \frac{3\pi}{4}, & x > 1. \end{cases}$$

Plot the sketch of graph next inverse trigonometric functions

$$181. y = \arcsin(2x+1).$$

$$182. y = \arccos(3x-2).$$

$$183. y = \operatorname{arctg}(2-3x).$$

$$184. y = \operatorname{arcctg}(1-2x).$$

$$185. y = \arcsin\left(\frac{1-5x}{4}\right).$$

$$186. y = \arccos\left(\frac{1+3x}{7}\right).$$

$$187. y = \arccos\left(\frac{1-|x|}{2}\right).$$

$$188. y = \arcsin\left(\frac{2+3|x|}{4}\right).$$

$$189. y = \operatorname{arctg}\left(\frac{1+|x|}{4}\right).$$

$$190. y = \operatorname{arcctg}\left(\frac{2|x|-3}{5}\right).$$

$$191. y = \arcsin \frac{1}{x+2}.$$

$$192. y = \arccos \frac{2}{x-3}.$$

$$193. y = \operatorname{arctg} \frac{1}{x}.$$

$$194. y = \operatorname{arcctg} \frac{1}{x}.$$

$$195. y = \operatorname{arctg} \frac{x+2}{x-3}.$$

$$196. y = \operatorname{arcctg} \frac{x+1}{|x|-2}.$$

$$197. y = \operatorname{arctg} \frac{|1-x|}{\sqrt{3}x+2}.$$

$$198. y = \arcsin \frac{1+x}{1-x}.$$

$$199. y = \frac{1}{\operatorname{arctg} ||x|-1|}.$$

$$200. y = \frac{1}{\arcsin \left| \frac{1-|x|}{3} \right|}.$$

$$201. y = \arcsin(\sin x).$$

$$202. y = \arcsin(\cos x).$$

$$203. y = \arccos(\cos x).$$

$$204. y = \arccos(\sin x).$$

$$205. y = \operatorname{arctg}(\operatorname{tg} x).$$

$$206. y = \operatorname{arcctg}(\operatorname{tg} x).$$

$$207. y = \arcsin(\operatorname{tg} x).$$

$$208. y = \arccos(\operatorname{ctg} x).$$

$$209. y = \sin \arcsin 2x.$$

$$210. y = \cos \left(\arccos \frac{1}{x} \right).$$

$$211. y = \sin(\operatorname{arctg} x).$$

$$212. y = \sin(\operatorname{arcctg} x).$$

$$213. y = \operatorname{tg}(\arcsin x).$$

$$214. y = \operatorname{ctg}(\operatorname{arctg} x).$$

$$215. \quad y = \operatorname{tg}(\arccos x).$$

$$216. \quad y = \operatorname{ctg}(\arcsin x).$$

$$217. \quad y = \arccos \sin x^3.$$

$$218. \quad y = \arcsin \cos \sqrt{x}.$$

$$219. \quad y = \cos \arcsin \frac{1}{x}.$$

$$220. \quad y = \operatorname{arctg}(\operatorname{tg}(2x + 1)).$$

$$221. \quad y = \arcsin \frac{x^2 - 1}{x^2 + 1}.$$

$$222. \quad y = \operatorname{arctg} \frac{x^2 - 1}{x^2 - 4}.$$

$$223. \quad y = \operatorname{arcctg} \frac{x^2 + 1}{x}.$$

$$224. \quad y = \operatorname{arctg} \frac{2^x + 1}{2^x - 1}.$$

$$225. \quad y = \arcsin \frac{(x-1)(x+2)^3}{(x+1)^3}.$$

$$226. \quad y = \arccos \frac{x^3 - 4x}{(x-1)^2(x+1)}.$$

$$227. \quad y = \operatorname{arctg} \frac{x^3 + 4x^2 + 4x}{x^2 - 9}.$$

$$228. \quad y = \operatorname{arcctg} \frac{x^4 - 4x^3 + 2x^2}{(x-2)^2(x+1)^2}.$$

$$229. \quad y = \operatorname{arctg} \frac{(x^3 - 1)(x+4)|x|}{(x^3 + 1)(x-3)^5}.$$

$$230. \quad y = \operatorname{arcctg} \frac{x^4 - 9x^2}{(x-4)^2(x+1)^3}.$$

$$231. \quad y = \operatorname{arctg} \frac{x^3 - x}{(x+2)^2(x-10)}.$$

$$232. \quad y = \operatorname{arcctg} \frac{(x+1)^2(x-2)^3}{(x-3)^2(x^2 + 1)}.$$