

Lesson 29 (even problems must be solved in class, odd examples must be solved at home)

36. Show that at each point of the lemniscate $\rho^2 = a^2 \cos 2\varphi$ the curvature is proportional to the radius vector of the point.

37. Find the greatest value of the radius of curvature of the curve $\rho = a \sin^3 \frac{\varphi}{3}$. *Ans.* $R = \frac{3}{4} a$.

38. Find the coordinates of the centre of curvature of the curve $y = x \ln x$ at the point where $y' = 0$. *Ans.* $(e^{-1}, 0)$.

39. Prove that for points of the spiral of Archimedes $\rho = a\varphi$ as $\varphi \rightarrow \infty$ the magnitude of the difference between the radius vector and the radius of curvature approaches zero.

40. Find the parabola $y = ax^2 + bx + c$, which has common tangent and curvature with the sine curve $y = \sin x$ at the point $(\frac{\pi}{2}, 1)$. Make a drawing.

Ans. $y = -\frac{x^2}{2} + \frac{\pi x}{2} + 1 - \frac{\pi^2}{8}$.

41. The function $y = f(x)$ is defined as follows:

$$f(x) = x^3 \text{ in the interval } -\infty < x \leq 1$$

$$f(x) = ax^2 + bx + c \text{ in the interval } 1 < x < +\infty$$

What must a , b and c be for the curve $y = f(x)$ to have continuous curvature everywhere? Make a drawing. *Ans.* $a = 3$, $b = -3$, $c = 1$.

42. Show that the radius of curvature of a cycloid at any one of its points is twice the length of the normal at that point.

43. Write the equation of the circle of curvature of the parabola $y = x^2$ at the point $(1, 1)$. *Ans.* $(x+4)^2 + (y - \frac{7}{2})^2 = \frac{125}{4}$.

44. Write the equation of the circle of curvature of the curve $y = \tan x$ at the point $(\frac{\pi}{4}, 1)$. *Ans.* $(x - \frac{\pi-10}{4})^2 + (y - \frac{9}{4})^2 = \frac{125}{16}$.

45. Find the length of the entire evolute of an ellipse whose semi-axes are a and b . *Ans.* $\frac{4(a^3 - b^3)}{ab}$.

46. Find the approximate values of the roots of the equation $xe^x = 2$ to within 0.01. *Ans.* The equation has only one real root, $x \approx 0.84$.

47. Find the approximate values of the roots of the equation $x \ln x = 0.8$ to within 0.01. *Ans.* The equation has only one real root, $x \approx 1.64$.

48. Find the approximate values of the roots of the equation $x^2 \arctan x = 1$ to within 0.001. *Ans.* The equation has only one real root, $x \approx 1.096$.