Lesson 28 (even problems must be solved in class, odd examples must be solved at home)

Find the curvature of the curves at the indicated points:

1. $b^2x^2+a^2y^2=a^2b^2$ at the points $(0, b)$ and $(a, 0)$. *Ans.* $\frac{b}{a^2}$ at $(0, b)$; $\frac{a}{b^2}$ at $(a, 0)$. 2. $xy=12$ at the point $(3, 4)$. *Ans.* $\frac{24}{125}$. 3. $y=x^3$ at the point (x_1, y_1). *Ans.* $\frac{6x_1}{(1+9x_1^4)^{3/2}}$. 4. $16y^2=4x^4-x^6$ at the point $(2, 0)$. *Ans.* $\frac{1}{2}$.

5. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ at an arbitrary point. *Ans.* $\frac{1}{3\sqrt[3]{|axy|}}$.

Find the radius of curvature of the following curves at the indicated points; draw each curve and construct the appropriate circle of curvature:

6. $y^2=x^3$ at the point $(4, 8)$. *Ans.* $R=\frac{80\sqrt{10}}{3}$. 7. $x^2=4ay$ at the point $(0, 0)$. *Ans.* $R=2a$. 8. $b^2x^2-a^2y^2=a^2b^2$ at the point (x_1, y_1). *Ans.* $R=\frac{(b^4x_1+a^4y_1)^{3/2}}{a^4b^4}$.

9. $y=\ln x$ at the point $(1, 0)$. *Ans.* $R=2\sqrt{2}$. 10. $y=\sin x$ at the point $\left(\frac{\pi}{2}, 1\right)$. *Ans.* $R=1$. 11. $\left.\begin{aligned}x&=a\cos^3 t\\ y&=a\sin^3 t\end{aligned}\right\}$ for $t=t_1$. *Ans.* $R=3a\sin t_1\cos t_1$.

Find the radius of curvature of the indicated curves:

12. $\left.\begin{aligned}x&=3t^2\\ y&=3t-t^3\end{aligned}\right\}$ for $t=1$. *Ans.* $R=6$. 13. Circle $\rho=a\sin\theta$. *Ans.* $R=\frac{a}{2}$.

14. Spiral of Archimedes $\rho=a\theta$. *Ans.* $R=\frac{(\rho^2+a^2)^{3/2}}{\rho^2+2a^2}$. 15. Cardioid $\rho=a(1-\cos\theta)$. *Ans.* $R=\frac{2}{3}\sqrt{2a\rho}$. 16. Lemniscate $\rho^2=a^2\cos 2\theta$. *Ans.* $R=\frac{a^2}{3\rho}$.

17. Parabola $\rho=a\sec^2\frac{\theta}{2}$. *Ans.* $R=2a\sec^3\frac{\theta}{2}$. 18. $\rho=a\sin^3\frac{\theta}{3}$. *Ans.* $R=\frac{3}{4}a\sin^2\frac{\theta}{3}$.

Find the points of the curves at which the radius of curvature is a minimum:

19. $y=\ln x$. *Ans.* $\left(\frac{\sqrt{2}}{2}, -\frac{1}{2}\ln 2\right)$. 20. $y=e^x$. *Ans.* $\left(-\frac{1}{2}\ln 2, \frac{\sqrt{2}}{2}\right)$.

21. $\sqrt{x}+\sqrt{y}=\sqrt{a}$. *Ans.* $\left(\frac{a}{4}, \frac{a}{4}\right)$. 22. $y=a\ln\left(1-\frac{x^2}{a^2}\right)$. *Ans.* At the point $(0, 0)$ $R=\frac{a}{2}$.

Find the coordinates of the centre of curvature (α, β) and the equation of the evolute for each of the following curves:

23. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. *Ans.* $\alpha=\frac{(a^2+b^2)x^3}{a^4}$, $\beta=-\frac{(a^2+b^2)y^3}{b^4}$. 24. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. *Ans.* $\alpha=x+3x^{\frac{1}{3}}y^{\frac{2}{3}}$, $\beta=y+3x^{\frac{2}{3}}y^{\frac{1}{3}}$. 25. $y^3=a^2x$. *Ans.* $\alpha=\frac{a^4+15y^4}{6a^2y}$, $\beta=\frac{a^4y-9y^5}{2a^4}$. 26. $\left\{\begin{aligned}x&=3t,\\ y&=t^2-6.\end{aligned}\right.$ *Ans.* $\alpha=-\frac{4}{3}t^3$, $\beta=3t^2-\frac{3}{2}$.

27. $\begin{cases} x = k \ln \cot \dfrac{t}{2} - k \cos t, \\ y = k \sin t. \end{cases}$ *Ans.* $y = \dfrac{k}{2}\left(e^{\frac{x}{k}} + e^{-\frac{x}{k}}\right)$ (tractrix).

28. $\begin{cases} x = a(\cos t + t \sin t), \\ y = a(\sin t - t \cos t). \end{cases}$ *Ans.* $\alpha = a \cos t$, $\beta = a \sin t$. 29. $\begin{cases} x = a \cos^3 t, \\ y = a \sin^3 t. \end{cases}$

Ans. $\alpha = a \cos^3 t + 3a \cos t \sin^2 t$, $\beta = a \sin^3 t + 3a \cos^2 t \sin t$.

30. Find the roots of the equation $x^3 - 4x + 2 = 0$ to three decimal places. *Ans.* $x_1 = 1.675$, $x_2 = 0.539$, $x_3 = -2.214$.

31. For the equation $f(x) = x^5 - x - 0.2 = 0$, approximate the root in the interval (1, 1.1). *Ans.* 1.045.

32. Evaluate the roots of the equation $x^4 + 2x^2 - 6x + 2 = 0$ to two decimal places. *Ans.* $0.38 < x_1 < 0.39$, $1.24 < x_2 < 1.25$.

33. Solve the equation $x^3 - 5 = 0$ approximately. *Ans.* $x_1 \approx 1.71$, $x_{2,3} = 1.71 \dfrac{-1 \pm i\sqrt{3}}{2}$.

34. Approximate the root of the equation $x - \tan x = 0$ lying between 0 and $\dfrac{3\pi}{2}$. *Ans.* 4.4935.

35. Compute the root of the equation $\sin x = 1 - x$ to three places of decimals. **Hint.** Reduce the equation to the form $f(x) = 0$. *Ans.* $0.5110 < x < 0.5111$.