Lesson 20 (even problems must be solved in class, odd examples must be solved at home)

Rolle's Theorem. *If a function $f(x)$ is continuous on an interval $[a, b]$ and is differentiable at ail interior points of the interval, and vanishes $[f(a) = f(b) = 0]$ at the end points $x = a$ and $x = b$, then inside $[a, b]$ there exists at least one point $x = c$, $a < c < b$, at which the derivative $f(x)$ vanishes, that is, $f'(c) = 0$.*

Verify the truth of Rolle's theorem for the functions: **1.** $y = x^2 - 3x + 2$ on the interval $[1, 2]$. **2.** $y = x^3 + 5x^2 - 6x$ on the interval $[0, 1]$. **3.** $y = (x-1) \times \times (x-2)(x-3)$ on the interval $[1, 3]$. **4.** $y = \sin^2 x$ on the interval $[0, \pi]$.

5. The function $f(x) = 4x^3 + x^2 - 4x - 1$ has roots 1 and -1. Find the root of the derivative $f'(x)$ mentioned in Rolle's theorem.

6. Verify that between the roots of the function $y = \sqrt[3]{x^2 - 5x + 6}$ lies the root of its derivative.

7. Verify the truth of Rolle's theorem for the function $y = \cos^2 x$ on the interval $\left[-\frac{\pi}{4}, +\frac{\pi}{4}\right]$.

8. The function $y = 1 - \sqrt[5]{x^4}$ becomes zero at the end points of the interval $[-1, 1]$. Make it clear that the derivative of this function does not vanish anywhere in the interval $(-1, 1)$. Explain why Rolle's theorem is not applicable here.

Lagrange's Theorem. *If a function $f(x)$ is continuous on the interval $[a, b]$ and differentiable at all interior points of the internal, there will be, within $[a, b]$, at least one point c, $a < c < b$, such that*

$$f(b) - f(a) = f'(c)(b - a).$$

9. Form Lagrange's formula for the function $y = \sin x$ on the interval $[x_1, x_2]$. *Ans.* $\sin x_2 - \sin x_1 = (x_2 - x_1)\cos c$, $x_1 < c < x_2$.

10. Verify the truth of the Lagrange formula for the function $y = 2x - x^2$ on the interval $[0, 1]$.

11. At what point is the tangent to the curve $y = x^n$ parallel to the chord from point $M_1(0, 0)$ to $M_2(a, a^n)$? *Ans.* At the point with abscissa $c = \frac{a}{\sqrt[n-1]{n}}$.

12. At what point is the tangent to the curve $y = \ln x$ parallel to the chord linking the points $M_1(1, 0)$ and $M_2(e, 1)$? *Ans.* At the point with abscissa $c = e - 1$.

Applying the Lagrange theorem, prove the inequalities: **13.** $e^x \geqslant 1 + x$. **14.** $\ln(1 + x) < x$ $(x > 0)$. **15.** $b^n - a^n < nb^{n-1}(b - a)$ for $(b > a)$. **16.** $\arctan x < x$.

17. Write the Cauchy formula for the functions $f(x) = x^2$, $\varphi(x) = x^3$ on the interval $[1, 2]$ and find c. *Ans.* $c = \frac{14}{9}$.