Lesson 2

ALGEBRAIC FUNCTIONS

Algebraic functions include elementary functions of the following kind:

I. The rational integral function, or **polynomial**

$$y = a_0x^n + a_1x^{n-1} + \ldots + a_n$$

where a_0, a_1, ..., a_n are constants called *coefficients* and n is a nonnegative integer called the *degree of the polynomial.* It is obvious that this function is defined for all values of x, that is, it is defined in an infinite interval.

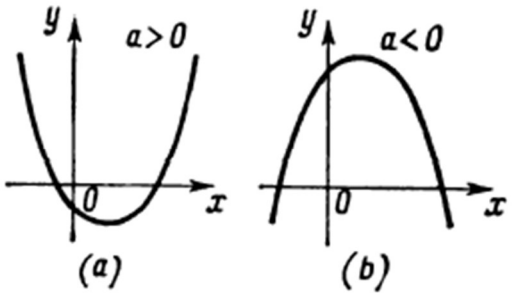

Fig. 21

Examples. 1. $y = ax + b$ is a *linear function.* When $b = 0$, the linear function $y = ax$ expresses y as being directly proportional to x. For $a = 0$, $y = b$, the function is a constant.

2. $y = ax^2 + bx + c$ is a *quadratic function.* The graph of a quadratic function is a parabola (Fig. 21). These functions are considered in detail in analytic geometry.

II. Fractional rational function. This function is defined as the ratio of two polynomials:

$$y = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_m}$$

For example, the following is a fractional rational function:

$$y = \frac{a}{x}$$

It expresses inverse variation. Its graph is shown in Fig. 22. It is obvious that a fractional rational function is defined for all values of x with the exception of those for which the denominator becomes zero.

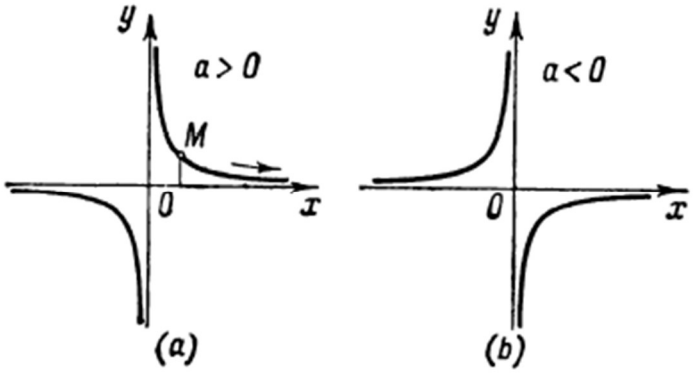

Fig. 22

III. Irrational function. If in the formula $y = f(x)$, operations of addition, subtraction, multiplication, division and raising to a power with rational nonintegral exponents are performed on the right-hand side, the function $y = f(x)$ is *irrational.* Examples of ir-

rational functions are: $y=\dfrac{2x^2+\sqrt{x}}{\sqrt{1+5x^2}}$, $y=\sqrt{x}$, etc.

Note 1. The three types of algebraic functions mentioned above do not exhaust all algebraic functions. An *algebraic function* is any function $y=f(x)$ which satisfies an equation of the form

$$P_0(x)\,y^n+P_1(x)\,y^{n-1}+\ldots+P_n(x)=0 \qquad (1)$$

where $P_0(x)$, $P_1(x)$, ..., $P_n(x)$ are certain polynomials in x.

It may be proved that each of these three types of function satisfies a certain equation of type (1), but not every function that satisfies an equation like (1) is a function of one of the three types given above.

Note 2. A function which is not algebraic is called *transcendental*. Examples of transcendental functions are $y=\cos x$, $y=10^x$ and the like.

1. Given the function $f(x)=x^2+6x-4$. Verify that $f(1)=3$, $f(3)=23$.

2. $f(x)=x^2+1$. Evaluate: (a) $f(4)$. *Ans.* 17. (b) $f(\sqrt{2})$. *Ans.* 3. (c) $f(a+1)$. *Ans.* a^2+2a+2. (d) $f(a)+1$. *Ans.* a^2+2. (e) $f(a^2)$. *Ans.* a^4+1. (f) $[f(a)]^2$. *Ans.* a^4+2a^2+1. (g) $f(2a)$. *Ans.* $4a^2+1$.

3. $\varphi(x)=\dfrac{x-1}{3x+5}$. Write the expressions $\varphi\left(\dfrac{1}{x}\right)$ and $\dfrac{1}{\varphi(x)}$. *Ans.* $\varphi\left(\dfrac{1}{x}\right)=\dfrac{1-x}{3+5x}$, $\dfrac{1}{\varphi(x)}=\dfrac{3x+5}{x-1}$.

4. $\psi(x)=\sqrt{x^2+4}$. Write the expressions $\psi(2x)$ and $\psi(0)$. *Ans.* $\psi(2x)=2\sqrt{x^2+1}$, $\psi(0)=2$.

5. $f(\theta)=\tan\theta$. Verify the equation $f(20)=\dfrac{2f(\theta)}{1-[f(\theta)]^2}$.

6. $\varphi(x)=\log\dfrac{1-x}{1+x}$. Verify the equation $\varphi(a)+\varphi(b)=\varphi\left(\dfrac{a+b}{1+ab}\right)$.

7. $f(x)=\log x$; $\varphi(x)=x^3$. Write the expressions:

(a) $f[\varphi(2)]$. *Ans.* $3\log 2$. (b) $f[\varphi(a)]$. *Ans.* $3\log a$.

(c) $\varphi[f(a)]$. *Ans.* $[\log a]^3$.

8. Find the natural domain of definition of the function $y=2x^2+1$. *Ans.* $-\infty<x<+\infty$.

9. Find the natural domains of definition of the functions: (a) $\sqrt{1-x^2}$. *Ans.* $-1\leqslant x\leqslant+1$. (b) $\sqrt{3+x}+\sqrt[4]{7-x}$. *Ans.* $-3\leqslant x\leqslant 7$. (c) $\sqrt[3]{x+a}-\sqrt[5]{x-b}$. *Ans.* $-\infty<x<+\infty$. (d) $\dfrac{a+x}{a-x}$. *Ans.* $x\neq a$. (e) $\arcsin^2 x$. *Ans.* $-1\leqslant x\leqslant 1$. (f) $y=\log x$. *Ans.* $x>0$. (g) $y=a^x$ $(a>0)$. *Ans.* $-\infty<x<+\infty$.

Construct the graphs of the functions:

10. $y=-3x+5$. **11.** $y=\dfrac{1}{2}x^2+1$. **12.** $y=3-2x^2$. **13.** $y=x^2+2x-1$.

14. $y=\dfrac{1}{x-1}$. **15.** $y=\sin 2x$. **16.** $y=\cos 3x$. **17.** $y=x^2-4x+6$. **18.** $y=\dfrac{1}{1-x^2}$.

19. $y=\sin\left(x+\frac{\pi}{4}\right)$. **20.** $y=\cos\left(x-\frac{\pi}{3}\right)$. **21.** $y=\tan\frac{1}{2}x$. **22.** $y=\cot\frac{1}{4}x$.

23. $y=3^x$. **24.** $y=2^{-x^2}$. **25.** $y=\log_2\frac{1}{x}$. **26.** $y=x^3+1$. **27.** $y=4-x^3$.

28. $y=\frac{1}{x^2}$. **29.** $y=x^4$. **30.** $y=x^5$. **31.** $y=x^{\frac{1}{2}}$. **32.** $y=x^{-\frac{1}{2}}$. **33.** $y=x^{\frac{1}{3}}$.

34. $y=|x|$. **35.** $y=\log_2|x|$. **36.** $y=\log_2(1-x)$. **37.** $y=3\sin\left(2x+\frac{\pi}{3}\right)$.

38. $y=4\cos\left(x+\frac{\pi}{2}\right)$. **39.** The function $f(x)$ is defined on the interval $[-1, 1]$ as follows:

$$f(x)=1+x \quad \text{for} \quad -1\leqslant x\leqslant 0,$$
$$f(x)=1-2x \quad \text{for} \quad 0\leqslant x<1$$

40. The function $f(x)$ is defined on the interval $[0, 2]$ as follows:

$$f(x)=x^3 \quad \text{for} \quad 0\leqslant x\leqslant 1,$$
$$f(x)=x \quad \text{for} \quad 1\leqslant x\leqslant 2.$$

Plot the graphs of next functions:

123. $y=\sqrt{(x-1)^2(x-2)}$. **124.** $y=\sqrt[3]{(x+3)^5(x-2)^2(x+1)}$.

125. $y=\sqrt[3]{x^2}-x$. **126.** $y=(x+1)^{\frac{1}{2}}(x-1)^{\frac{1}{3}}(x-2)^2(x-3)^{\frac{2}{3}}$.

127. $y=\sqrt{x^2+9}+\sqrt{x^2-9}$. **128.** $y=x^{2/3}+\sqrt[5]{(x-1)^2}$.

129. $y=5\frac{\sqrt[3]{(x+2)^2(x-3)}}{\sqrt{(x+1)^2(x-2)^3}\cdot\sqrt{x+10}}$.

130. $y=\frac{\sqrt[5]{(x+4)^7(x-2)^3(x+1)}}{\sqrt{(x+2)^2(x+5)}}$.