Lesson 1

BASIC ELEMENTARY FUNCTIONS. ELEMENTARY FUNCTIONS

The basic elementary functions are the following analytically represented functions.

I. *Power function:* $y = x^\alpha$, where α is a real number.*

II. *General exponential function:* $y = a^x$, where a is a positive number not equal to unity.

III. *Logarithmic function:* $y = \log_a x$, where the logarithmic base a is a positive number not equal to unity.**

IV. *Trigonometric functions:* $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$, $y = \sec x$, $y = \csc x$.

V. *Inverse trigonometric functions:*

$$y = \arcsin x, \quad y = \arccos x, \quad y = \arctan x,$$
$$y = \operatorname{arccot} x, \quad y = \operatorname{arcsec} x, \quad y = \operatorname{arccsc} x.$$

Let us consider the domains of definition and the graphs of the basic elementary functions.

Power function $y = x^\alpha$.

1. α is a positive integer. The function is defined in the infinite interval $-\infty < x < +\infty$. In this case, the graphs of the function for certain values of α have the form shown in Figs. 6 and 7.

2. α is a negative integer. In this case, the function is defined for all values of x with the exception of $x = 0$. The graphs of the functions for certain values of α have the form shown in Figs. 8 and 9.

Figs. 10, 11, and 12 show graphs of a power function with fractional rational values of α.

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*If α is irrational, this function is evaluated by taking logarithms and antilogarithms: $\log y = \alpha \log x$. It is assumed here that $x > 0$.

**Throughout this book, the symbol log stands for the logarithm to the base 10.

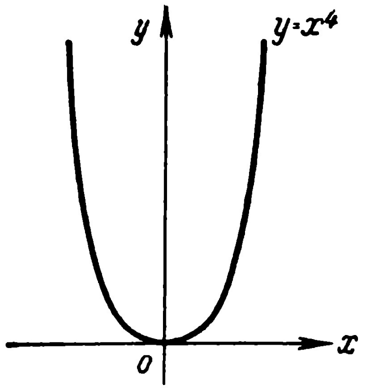


Fig. 6

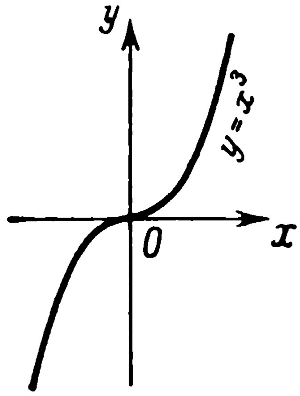


Fig. 7

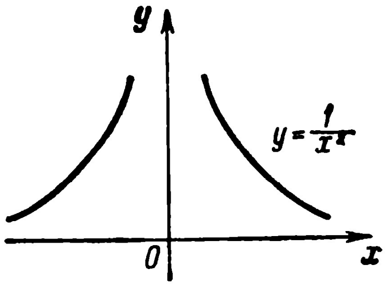


Fig. 8

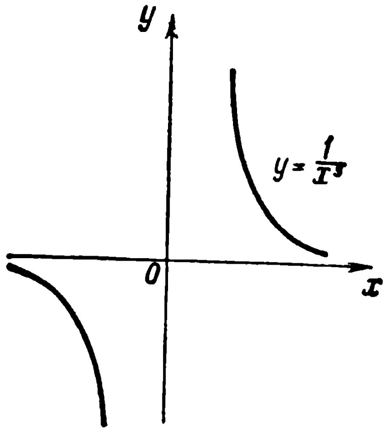


Fig. 9

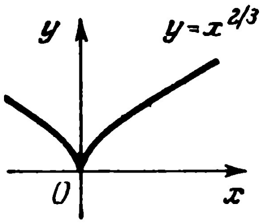
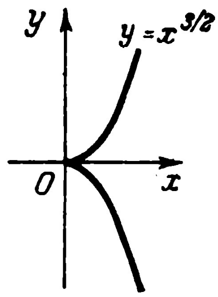


Fig. 10



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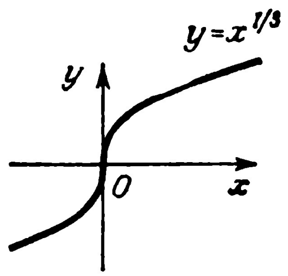

Fig. 12

General exponential function, $y=a^x$, $a>0$ and $a\neq 1$. This function is defined for all values of x. Its graph is shown in Fig. 13.

Logarithmic function, $y=\log_a x$, $a>0$ and $a\neq 1$. This function is defined for $x>0$. Its graph is shown in Fig. 14.

Trigonometric functions. In the formulas $y=\sin x$, etc. the independent variable x is expressed in radians. All the enumerated trigonometric functions are periodic. We give a general definition of a periodic function.

Definition 1. The function $y = f(x)$ is called *periodic* if there exists a constant C, which, when added to (or subtracted from) the argument x, does not change the value of the function:

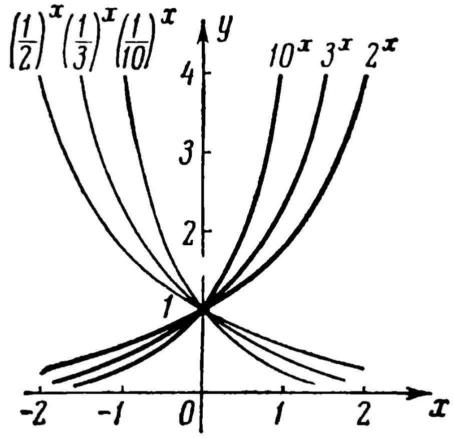

Fig. 13

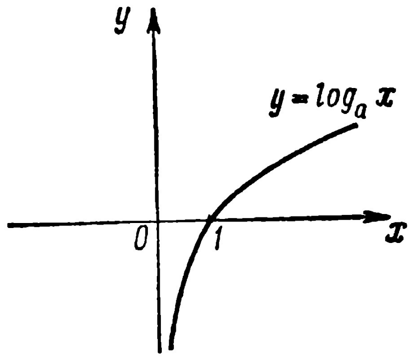

Fig. 14

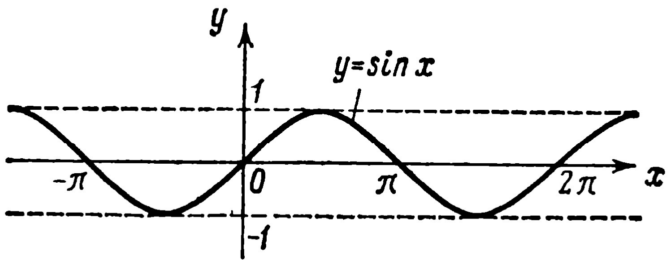

Fig. 15

$f(x + C) = f(x)$. The least such number is called the *period* of the function; it will henceforward be designated as $2l$.

From the definition it follows directly that $y = \sin x$ is a periodic function with period 2π: $\sin x = \sin(x + 2\pi)$. The period of $\cos x$

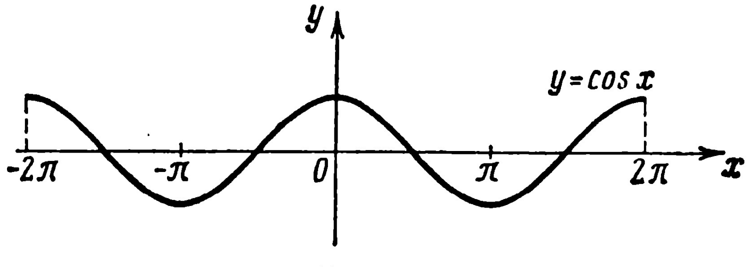

Fig. 16

is likewise 2π. The functions $y = \tan x$ and $y = \cot x$ have a period equal to π.

The functions $y = \sin x$, $y = \cos x$ are defined for all values of x; the functions $y = \tan x$ and $y = \sec x$ are defined everywhere except at the points $x = (2k + 1)\frac{\pi}{2}$ $(k = 0, \pm 1, \pm 2, \ldots)$; the functions

$y=\cot x$ and $y=\csc x$ are defined for all values of x except at the points $x=k\pi$ ($k=0, \pm 1, \pm 2, \ldots$). Graphs of trigonometric functions are shown in Figs. 15 to 19.

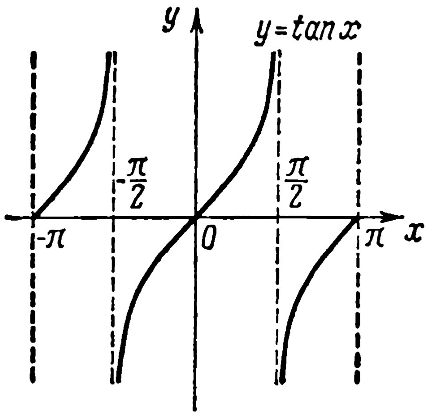

Fig. 17

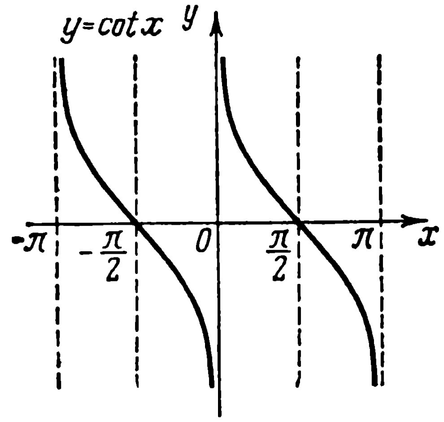

Fig. 18

The inverse trigonometric functions will be discussed in more detail later on.

Let us now introduce the concept of a function of a function. If y is a function of u, and u (in turn) is dependent on the variable x, then y is also dependent on x. Let $y=F(u)$ and $u=\varphi(x)$. We get y as a function of x:

$$y=F[\varphi(x)]$$

This function is called a *function of a function* or a *composite function*.

Example 1. Let $y=\sin u$, $u=x^2$. The function $y=\sin(x^2)$ is a composite function of x.

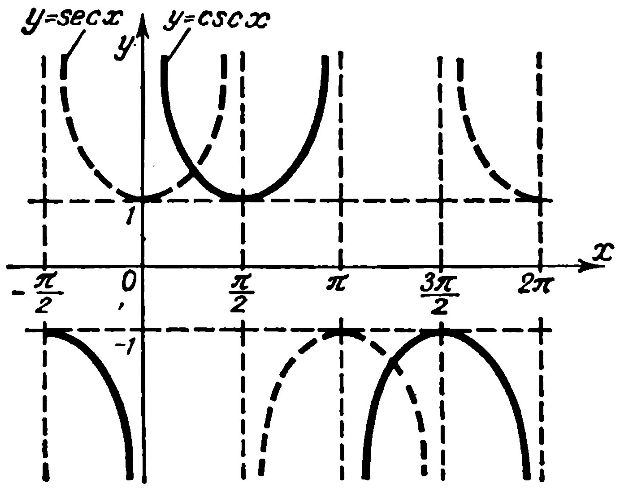

Fig. 19

Note. The domain of definition of the function $y=F[\varphi(x)]$ is either the entire domain of the function, $u=\varphi(x)$, or that part of it in which those values of u are defined that do not lie outside the domain of the function $F(u)$.

Example 2. The domain of definition of the function $y=\sqrt{1-x^2}$ ($y=\sqrt{u}$, $u=1-x^2$) is the closed interval $[-1, 1]$, because when $|x|>1$ $u<0$ and, consequently, the function $\sqrt{u}$ is not defined (although the function $u=1-x^2$ is defined for all values of x). The graph of this function is the upper half of a circle with centre at the origin of the coordinate system and with radius unity.

The operation "function of a function" may be performed any number of times. For instance, the function $y = \ln[\sin(x^2+1)]$ is obtained as a result of the following operations (defining the following functions);

$$v = x^2 + 1, \quad u = \sin v, \quad y = \ln u$$

Let us now define an elementary function.

Definition 2. An *elementary function* is a function which may be represented by a single formula of the type $y = f(x)$, where the expression on the right-hand side is made up of basic elementary functions and constants by means of a finite number of operations of addition, subtraction, multiplication, division and taking the function of a function.

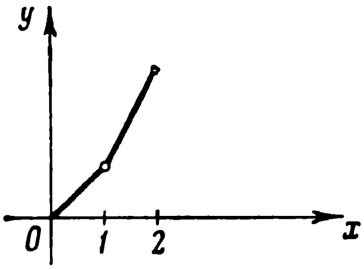

Fig. 20

From this definition it follows that elementary functions are functions represented analytically.

Examples of elementary functions:

$$y = |x| = \sqrt{x^2}, \quad y = \sqrt{1 + 4\sin^2 x}, \quad y = \frac{\log x + 4\sqrt[3]{x} + 2\tan x}{10^x - x + 10}$$

Example of nonelementary function:

The function $y = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n\ [y = f(n)]$ is not elementary because the number of operations that must be performed to obtain y increases with n, that is to say, it is not bounded.

Note. The function given in Fig. 20 is elementary even though it is represented by means of two formulas:

$$f(x) = x \quad \text{if} \quad 0 \leqslant x \leqslant 1$$
$$f(x) = 2x - 1 \quad \text{if} \quad 1 \leqslant x \leqslant 2$$

This function can also be defined by a single formula:

$$f(x) = \frac{3}{2}\left(x - \frac{1}{3}\right) + \frac{1}{2}|x - 1| = \frac{3}{2}\left(x - \frac{1}{3}\right) + \frac{1}{2}\sqrt{(x-1)^2}$$

for $0 \leqslant x \leqslant 2$.