PARAMETRIC REPRESENTATION OF A FUNCTION

Given two equations:

$$\left.\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \end{aligned}\right\} \qquad (1)$$

where t assumes values that lie in the interval $[T_1, T_2]$. To each value of t there correspond values of x and y (the functions φ and ψ are assumed to be single-valued). If one regards the values of x and y as coordinates of a point in a coordinate xy-plane, then to each value of t there will correspond a definite point in the plane. And when t varies from T_1 to T_2, this point will describe a certain curve. Equations (1) are called *parametric equations* of this curve, t is the *parameter*, and *parametric* is the way the curve is represented by equations (1).

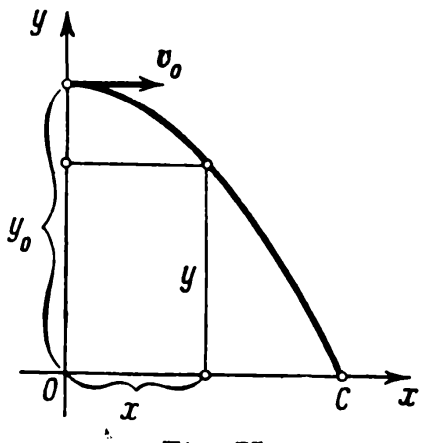

Fig. 75

Let us further assume that the function $x = \varphi(t)$ has an inverse, $t = \Phi(x)$. Then, obviously, y is a function of x;

$$y = \psi[\Phi(x)] \qquad (2)$$

Thus, equations (1) define y as a function of x, and we say that the function y of x is represented parametrically.

The explicit expression of the dependence of y on x, $y = f(x)$, is obtained by eliminating the parameter t from equations (1).

Parametric representation of curves is widely used in mechanics. If in the xy-plane there is a certain material point in motion and if we know the laws of motion of the projections of this point on the coordinate axes, then

$$\left.\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \end{aligned}\right\} \qquad (1')$$

where the parameter t is the time. Then equations (1′) are parametric equations of the trajectory of the moving point. Eliminating from these equations the parameter t, we get the equation of the trajectory in the form $y = f(x)$ or $F(x, y) = 0$. By way of illustration, let us take the following problem.

Problem. Determine the trajectory and point of impact of a load dropped from an airplane moving horizontally with a velocity v_0 at an altitude y_0 (air resistance is disregarded).

Solution. Taking a coordinate system as shown in Fig. 75, we assume that the airplane drops the load at the instant it cuts the y-axis. It is obvious that the horizontal translation of the load will be uniform and with constant velocity v_0:

$$x = v_0 t$$

Vertical displacement of the falling load due to the force of gravity will be expressed by the formula

$$s=\frac{gt^2}{2}$$

Hence the distance of the load from the ground at any instant will be

$$y=y_0-\frac{gt^2}{2}$$

The two equations

$$x=v_0t$$
$$y=y_0-\frac{gt^2}{2}$$

are the parametric equations of the trajectory. To eliminate the parameter t, we find the value $t=\frac{x}{v_0}$ from the first equation and substitute it into the second equation. Then we get the equation of the trajectory in the form

$$y=y_0-\frac{g}{2v_0^2}x^2$$

This is the equation of a parabola with vertex at the point $M(0, y_0)$, the y-axis serving as the axis of symmetry of the parabola.

We determine the length of OC. Denote the abscissa of C by X, and note that the ordinate of this point is $y=0$. Putting these values into the preceding formula, we get

$$0=y_0-\frac{g}{2v_0^2}X^2$$

whence

$$X=v_0\sqrt{\frac{2y_0}{g}}$$

3.17 THE EQUATIONS OF SOME CURVES IN PARAMETRIC FORM

Circle. Given a circle with centre at the coordinate origin and with radius r (Fig. 76).

Denote by t the angle formed by the x-axis and the radius to some point $M(x, y)$ of the circle. Then the coordinates of any point on the circle will be expressed in terms of the parameter t as follows:

$$\left.\begin{aligned}x&=r\cos t,\\ y&=r\sin t,\end{aligned}\right\}\ 0\leqslant t\leqslant 2\pi$$

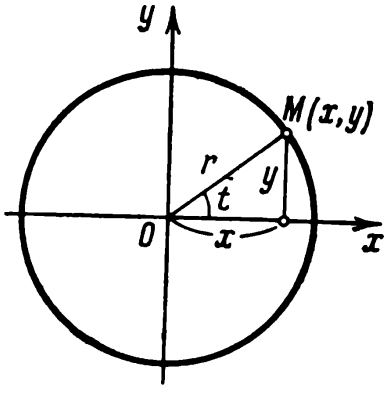

Fig. 76

These are the parametric equations of the circle. If we eliminate the parameter t from these equations, we will have an equation of the circle containing only x and y. Squaring the parametric equations and adding, we get

$$x^2+y^2=r^2(\cos^2 t+\sin^2 t)$$

or

$$x^2+y^2=r^2$$

Ellipse. Given the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

Set

$$x=a\cos t \tag{2'}$$

Putting this expression into equation (1) and performing the necessary manipulations, we get

$$y=b\sin t \tag{2''}$$

The equations

$$\left.\begin{aligned} x&=a\cos t,\\ y&=b\sin t,\end{aligned}\right\}\ 0\leqslant t\leqslant 2\pi \tag{2}$$

are the parametric equations of the ellipse.

Let us find out the geometrical meaning of the parameter t. Draw two circles with centres at the coordinate origin and with radii a and b (Fig. 77). Let the point M (x, y) lie on the ellipse, and let B be a point of the large circle with the same abscissas as M. Denote by t the angle formed by the radius OB with the x-axis. From the figure it follows directly that

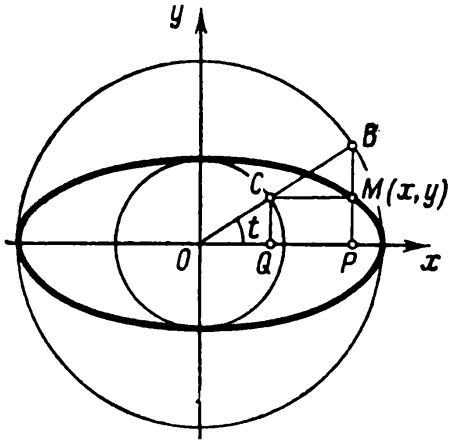

Fig. 77

$$\begin{aligned} x=OP&=a\cos t \\ CQ&=b\sin t\end{aligned} \tag{2'}$$

From (2'') we conclude that $CQ=y$; in other words, the straight line CM is parallel to the x-axis.

Consequently, in equations (2) t is an angle formed by the radius OB and the axis of abscissas. The angle t is sometimes called an *eccentric angle.*

Cycloid. The *cycloid* is a curve described by a point lying on the circumference of a circle if the circle rolls upon a straight line without sliding (Fig. 78). Suppose that when motion began the point M of the rolling circle lay at the origin. Let us determine the coordinates of M after the circle has turned through

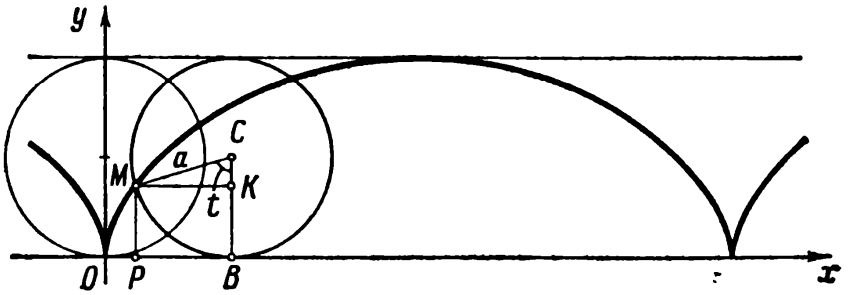

Fig. 78

an angle t. If a is the radius of the rolling circle, it will be seen from Fig. 78 that

$$x=OP=OB-PB$$

but since the circle rolls without sliding, we have

$$OB = \overset{\frown}{MB} = at, \quad PB = MK = a \sin t$$

Hence, $x = at - a \sin t = a(t - \sin t)$.

Further,

$$y = MP = KB = CB - CK = a - a \cos t = a(1 - \cos t)$$

The equations

$$\left.\begin{aligned} x &= a(t - \sin t) \\ y &= a(1 - \cos t) \end{aligned}\right\} 0 \leqslant t \leqslant 2\pi \tag{3}$$

are the parametric equations of the cycloid. As t varies between 0 and 2π, the point M will describe one arch of the cycloid.

Eliminating the parameter t from the latter equations, we get x as a function of y directly. In the interval $0 \leqslant t \leqslant \pi$, the function $y = a(1 - \cos t)$ has an inverse:

$$t = \arccos \frac{a - y}{a}$$

Substituting the expression for t into the first of equations (3), we get

$$x = a \arccos \frac{a - y}{a} - a \sin \left(\arccos \frac{a - y}{a} \right)$$

or

$$x = a \arccos \frac{a - y}{a} - \sqrt{2ay - y^2} \quad \text{when } 0 \leqslant x \leqslant \pi a$$

Examining the figure we note that when $\pi a \leqslant x \leqslant 2\pi a$

$$x = 2\pi a - \left(a \arccos \frac{a - y}{a} - \sqrt{2ay - y^2} \right)$$

It will be noted that the function

$$x = a(t - \sin t)$$

has an inverse, but it is not expressible in terms of elementary functions. And so the function $y = f(x)$ is not expressible in terms of elementary functions either.

Note 1. The cycloid clearly shows that in certain cases it is more convenient to use the parametric equations for studying functions and curves than the direct relationship of y and x (y as a function of x or x as a function of y).

Astroid. The *astroid* is a curve represented by the following parametric equations:

$$\left.\begin{aligned} x &= a \cos^3 t \\ y &= a \sin^3 t \end{aligned}\right\} 0 \leqslant t \leqslant 2\pi \tag{4}$$

Raising the terms of both equations to the power 2/3 and adding, we get the following relationship between x and y:

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} (\cos^2 t + \sin^2 t)$$

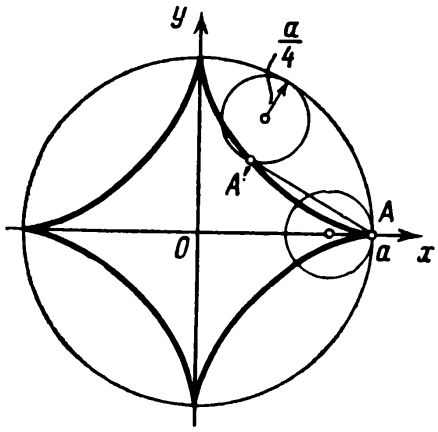

Fig. 79

or

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}} \tag{5}$$

Later on (Sec. 5.12) it will be shown that this curve is of the form shown in Fig. 79. It can be obtained as the trajectory of a certain point on the circumference of a circle of radius $a/4$ rolling (without sliding) upon another circle of radius a (the smaller circle always remains inside the larger one, see Fig. 79).

Note 2. It will be noted that equations (4) and equation (5) define more than one function $y=f(x)$. They define two continuous functions on the interval $-a \leqslant x \leqslant +a$. One takes on nonnegative values, the other nonpositive values.

THE DERIVATIVE OF A FUNCTION REPRESENTED PARAMETRICALLY

Let a function y of x be represented by the parametric equations

$$\left.\begin{aligned} x&=\varphi(t)\\ y&=\psi(t)\end{aligned}\right\} \quad t_0 \leqslant t \leqslant T \tag{1}$$

Let us assume that these functions have derivatives and that the function $x=\varphi(t)$ has an inverse, $t=\Phi(x)$, which also has a derivative. Then the function $y=f(x)$ defined by the parametric equations may be regarded as a composite function:

$$y=\psi(t), \quad t=\Phi(x)$$

t being the intermediate argument.

By the rule for differentiating a composite function we get

$$y'_x=y'_t t'_x=\psi'_t(t)\,\Phi'_x(x) \tag{2}$$

From the theorem for the differentiation of an inverse function, it follows that

$$\Phi'_x(x)=\frac{1}{\varphi'_t(t)}$$

Putting this expression into (2), we have

$$y'_x=\frac{\psi'(t)}{\varphi'(t)}$$

or

$$y'_x=\frac{y'_t}{x'_t} \tag{XXI}$$

The derived formula permits finding the derivative y'_x of a function represented parametrically without having to find y as a function of x.

Example 1. The function y of x is given by the parametric equations

$$\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}\right\} \quad (0 \leqslant t \leqslant \pi)$$

Find the derivative $\frac{dy}{dx}$: (1) for any value of t; (2) for $t=\frac{\pi}{4}$.

Solution.

(1) $y'_x = \frac{(a\sin t)'}{(a\cos t)'} = \frac{a\cos t}{-a\sin t} = -\cot t$;

(2) $(y'_x)_{t=\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1$.

Example 2. Find the slope of the tangent to the cycloid

$$x = a(t-\sin t)$$
$$y = a(1-\cos t)$$

at an arbitrary point $(0 \leqslant t \leqslant 2\pi)$.

Solution. The slope of the tangent at each point is equal to the value of the derivative y'_x at that point; i.e., it is

$$y'_x = \frac{y'_t}{x'_t}$$

But

$$x'_t = a(1-\cos t), \qquad y'_t = a\sin t$$

Consequently,

$$y'_x = \frac{a\sin t}{a(1-\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \cot\frac{t}{2} = \tan\left(\frac{\pi}{2}-\frac{t}{2}\right)$$

Hence, the slope of the tangent to a cycloid at every point is equal to $\tan\left(\frac{\pi}{2}-\frac{t}{2}\right)$, where t is the value of the parameter corresponding to this point. But this means that the angle α of inclination of the tangent to the x-axis is equal to $\frac{\pi}{2}-\frac{t}{2}$ (for values of t lying between $-\pi$ and π) *.

---

* Indeed, the slope is equal to the tangent of the angle of inclination α of the tangent to the x-axis. And so $\tan\alpha = \tan\left(\frac{\pi}{2}-\frac{t}{2}\right)$ and $\alpha = \frac{\pi}{2}-\frac{t}{2}$ for those values of t for which $\frac{\pi}{2}-\frac{t}{2}$ lies between 0 and π.

HYPERBOLIC FUNCTIONS

In many applications of mathematical analysis we encounter combinations of exponential functions of the form $\frac{1}{2}(e^x - e^{-x})$ and $\frac{1}{2}(e^x + e^{-x})$. These combinations are regarded as new functions and are designated as follows:

$$\left.\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \\ \cosh x &= \frac{e^x + e^{-x}}{2} \end{aligned}\right\} \qquad (1)$$

The first of these functions is called the *hyperbolic sine*, the second, the *hyperbolic cosine*. These functions may be used to define two more functions: $\tanh x = \frac{\sinh x}{\cosh x}$ and $\coth x = \frac{\cosh x}{\sinh x}$:

$$\left.\begin{aligned} \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}}, \text{ } \textit{the hyperbolic tangent,} \\ \coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}}, \text{ } \textit{the hyperbolic cotangent} \end{aligned}\right\} \qquad (1')$$

The functions $\sinh x$, $\cosh x$, $\tanh x$ are obviously defined for all values of x. But the function $\coth x$ is defined everywhere, except at the point $x = 0$. The graphs of the hyperbolic functions are given in Figs. 80, 81, 82.

From the definitions of the functions $\sinh x$ and $\cosh x$ [formulas (1)] there follow relationships similar to those between the appropriate trigonometric functions:

$$\cosh^2 x - \sinh^2 x = 1 \qquad (2)$$

$$\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b \qquad (3)$$

$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b \qquad (3')$$

Indeed,

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = 1$$

Further, noting that

$$\cosh(a+b) = \frac{e^{a+b} + e^{-a-b}}{2}$$

we get

$$\cosh a \cosh b + \sinh a \sinh b = \frac{e^a + e^{-a}}{2}\frac{e^b + e^{-b}}{2} + \frac{e^a - e^{-a}}{2}\frac{e^b - e^{-b}}{2}$$
$$= \frac{e^{a+b} + e^{-a+b} + e^{a-b} + e^{-a-b} + e^{a+b} - e^{-a+b} - e^{a-b} + e^{-a-b}}{4}$$
$$= \frac{e^{a+b} + e^{-a-b}}{2} = \cosh(a+b)$$

The proof is similar for relation (3′).

The name "hyperbolic functions" comes from the fact that the functions sinh t and cosh t play the same role in the parametric representation of the hyperbola.

$$x^2 - y^2 = 1$$

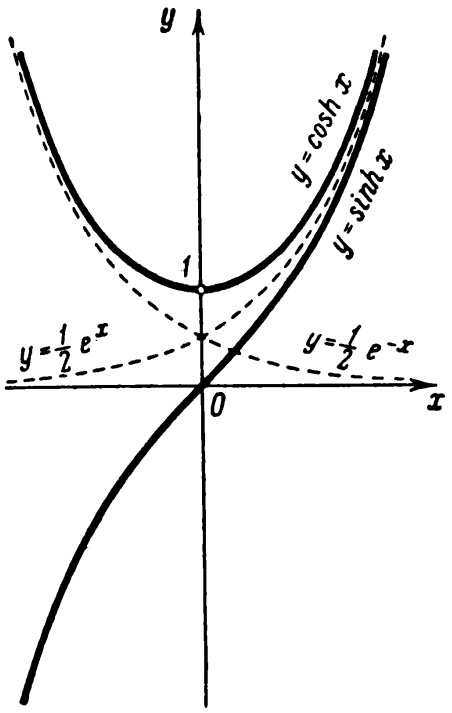

Fig. 80

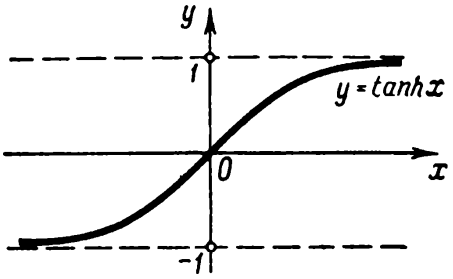

Fig. 81

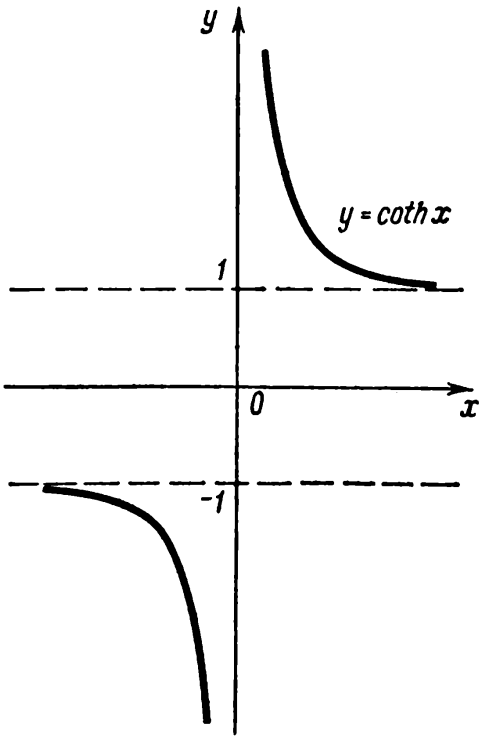

Fig. 82

as the trigonometric functions sin t and cos t do in the parametric representation of the circle

$$x^2 + y^2 = 1$$

Indeed, eliminating the parameter t from the equations

$$x = \cos t, \quad y = \sin t$$

we get

$$x^2+y^2=\cos^2 t+\sin^2 t$$

or

$$x^2+y^2=1 \text{ (the equation of the circle)}$$

Similarly, the equations

$$x=\cosh t$$
$$y=\sinh t$$

are the parametric equations of the hyperbola.

Indeed, squaring these equations termwise and subtracting the second from the first, we get

$$x^2-y^2=\cosh^2 t-\sinh^2 t$$

Since, on the basis of formula (2), the expression on the right is equal to unity, we have

$$x^2-y^2=1$$

which is the equation of the hyperbola.

Let us consider a circle with the equation $x^2+y^2=1$ (Fig. 83). In the equations $x=\cos t$, $y=\sin t$, the parameter t is numerically

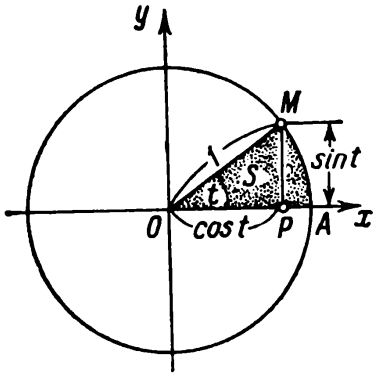

Fig. 83

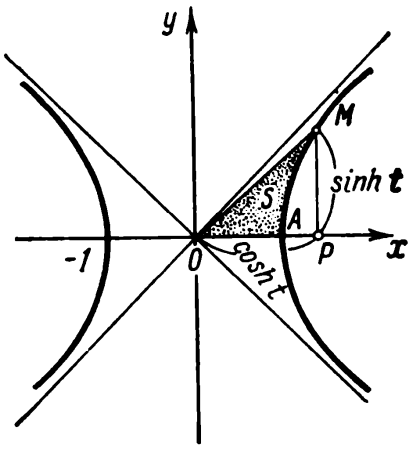

Fig. 84

equal to the central angle AOM or to the doubled area S of the sector AOM, since $t=2S$.

Let it be noted, without proof, that in the parametric equations of the hyperbola,

$$x=\cosh t$$
$$y=\sinh t$$

the parameter t is also numerically equal to twice the area of the "hyperbolic sector" AOM (Fig. 84).

The derivatives of the hyperbolic functions are defined by the formulas

$$\begin{aligned} (\sinh x)' &= \cosh x, \quad (\tanh x)' = \frac{1}{\cosh^2 x}, \\ (\cosh x)' &= \sinh x, \quad (\coth x)' = -\frac{1}{\sinh^2 x} \end{aligned} \tag{XXII}$$

which follow from the very definition of hyperbolic functions; for instance, for the function $\sinh x = \frac{e^x - e^{-x}}{2}$ we have

$$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x + e^{-x}}{2} = \cosh x$$

THE DIFFERENTIAL

Let a function $y = f(x)$ be differentiable on an interval $[a, b]$. The derivative of this function at some point x of $[a, b]$ is determined by the equation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

As $\Delta x \longrightarrow 0$, the ratio $\frac{\Delta y}{\Delta x}$ approaches a definite number $f'(x)$ and, consequently, differs from the derivative $f'(x)$ by an infinitesimal:

$$\frac{\Delta y}{\Delta x} = f'(x) + \alpha$$

where $\alpha \longrightarrow 0$ as $\Delta x \longrightarrow 0$.

Multiplying all terms by Δx, we get

$$\Delta y = f'(x)\,\Delta x + \alpha \Delta x \tag{1}$$

Since in the general case $f'(x) \neq 0$, for a constant x and a variable $\Delta x \longrightarrow 0$, the product $f'(x)\,\Delta x$ is an infinitesimal of the first order relative to Δx. But the product $\alpha \Delta x$ is always an infinitesimal of higher order than Δx because

$$\lim_{\Delta x \to 0} \frac{\alpha \Delta x}{\Delta x} = \lim_{\Delta x \to 0} \alpha = 0$$

Thus, the increment Δy of the function consists of two terms, of which the first is [when $f'(x) \neq 0$] the so-called **principal part** of the increment, and is **linear** in Δx. The product $f'(x)\,\Delta x$ is called the **differential** of the function and is denoted by dy or $df(x)$.

And so if a function $y = f(x)$ has a derivative $f'(x)$ at the point x, the product of the derivative $f'(x)$ by the increment Δx in the argument is called the *differential* of the function and is denoted by the symbol dy:

$$dy = f'(x)\,\Delta x \tag{2}$$

Find the differential of the function $y=x$; here,

$$y'=(x)'=1$$

and, consequently, $dy=dx=\Delta x$ or $dx=\Delta x$. Thus, **the differential dx of the independent variable x coincides with its increment Δx.** The equation $dx=\Delta x$ might be regarded likewise as a definition of the differential of the independent variable, and then the foregoing example would indicate that this does not contradict the definition of the differential of the function. In any case, we can write formula (2) as

$$dy=f'(x)\,dx$$

But from this relationship it follows that

$$f'(x)=\frac{dy}{dx}$$

Hence, **the derivative $f'(x)$ may be regarded as the ratio of the differential of the function to the differential of the independent variable.**

Let us return to expression (1), which, taking (2) into account, may be rewritten thus:

$$\Delta y=dy+\alpha\Delta x \tag{3}$$

Thus, the increment of a function differs from the differential of a function by an infinitesimal of higher order than Δx. If $f'(x)\neq 0$, then $\alpha\Delta x$ is an infinitesimal of higher order than dy and

$$\lim_{\Delta x\to 0}\frac{\Delta y}{dy}=1+\lim_{\Delta x\to 0}\frac{\alpha\Delta x}{f'(x)\,\Delta x}=1+\lim_{\Delta x\to 0}\frac{\alpha}{f'(x)}=1$$

For this reason, in approximate calculations one sometimes uses the approximate equation

$$\Delta y\approx dy \tag{4}$$

or, in expanded form,

$$f(x+\Delta x)-f(x)\approx f'(x)\,\Delta x \tag{5}$$

thus reducing the amount of computation.

Example 1. Find the differential dy and the increment Δy of the function $y=x^2$:

(1) for arbitrary values of x and Δx,

(2) for $x=20$, $\Delta x=0.1$.

Solution. (1) $\Delta y=(x+\Delta x)^2-x^2=2x\Delta x+\Delta x^2$,

$$dy=(x^2)'\,\Delta x=2x\Delta x.$$

(2) If $x=20$, $\Delta x=0.1$, then $\Delta y=2\cdot 20\cdot 0.1+(0.1)^2=4.01$,

$$dy=2\cdot 20\cdot 0.1=4.00.$$

Replacing Δy by dy yields an error of 0.01. In many cases, it may be considered small compared with $\Delta y = 4.01$ and therefore disregarded.

Fig. 85 gives a clear picture of the above problem.

In approximate calculations, one also makes use of the following equation, which is obtained from (5):

$$f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x \qquad (6)$$

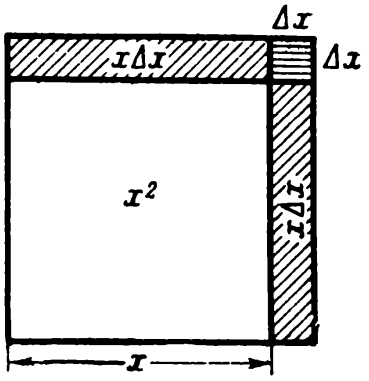

Fig. 85

Example 2. Let $f(x) = \sin x$, then $f'(x) = \cos x$. In this case the approximate equation (6) takes the form

$$\sin(x+\Delta x) \approx \sin x + \cos x\,\Delta x \qquad (7)$$

Let us calculate the approximate value of $\sin 46°$. Put $x = 45° = \frac{\pi}{4}$, $\Delta x = 1° = \frac{\pi}{180}$, $x + \Delta x = \frac{\pi}{4} + \frac{\pi}{180}$. Substituting into (7) we get

$$\sin 46° = \sin\left(\frac{\pi}{4} + \frac{\pi}{180}\right) \approx \sin\frac{\pi}{4} + \cos\frac{\pi}{4}\,\frac{\pi}{180}$$

or

$$\sin 46° \approx \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,\frac{\pi}{180} = 0.7071 + 0.7071 \cdot 0.0175 = 0.7191$$

Example 3. If in (7) we put $x = 0$, $\Delta x = \alpha$, we get the following approximate equation:

$$\sin\alpha \approx \alpha$$

Example 4. If $f(x) = \tan x$, then by (6) we get the following approximate equation:

$$\tan(x+\Delta x) \approx \tan x + \frac{1}{\cos^2 x}\,\Delta x$$

for $x = 0$, $\Delta x = \alpha$, we get

$$\tan\alpha \approx \alpha$$

Example 5. If $f(x) = \sqrt{x}$, then (6) yields

$$\sqrt{x+\Delta x} \approx \sqrt{x} + \frac{1}{2\sqrt{x}}\,\Delta x$$

Putting $x = 1$, $\Delta x = \alpha$, we get the approximate equation

$$\sqrt{1+\alpha} \approx 1 + \frac{1}{2}\alpha$$

The problem of finding the differential of a function is equivalent to that of finding the derivative, since, by multiplying the latter into the differential of the argument we get the differential of the function. Consequently, most theorems and formulas pertaining to derivatives are also valid for differentials. Let us illustrate this.

The differential of the sum of two differentiable functions u and v is equal to the sum of the differentials of these functions:

$$d(u+v) = du + dv$$

The differential of the product of two differentiable functions u and v is determined by the formula

$$d(uv) = u\,dv + v\,du$$

By way of illustration, let us prove the latter formula. If $y = uv$, then

$$dy = y'\,dx = (uv' + vu')\,dx = uv'\,dx + vu'\,dx$$

but

$$v'\,dx = dv, \quad u'\,dx = du$$

therefore

$$dy = u\,dv + v\,du$$

Other formulas (for instance, the formula defining the differential of a quotient) are proved in similar fashion:

$$\text{if} \quad y = \frac{u}{v}, \quad \text{then} \quad dy = \frac{v\,du - u\,dv}{v^2}$$

Let us solve some examples in calculating the differential of a function.

Example 6. $y = \tan^2 x$, $dy = 2 \tan x \dfrac{1}{\cos^2 x}\,dx$.

Example 7. $y = \sqrt{1 + \ln x}$, $dy = \dfrac{1}{2\sqrt{1 + \ln x}} \cdot \dfrac{1}{x}\,dx$.

We find the expression for the differential of a composite function. Let

$$y = f(u), \quad u = \varphi(x), \quad \text{or} \quad y = f[\varphi(x)]$$

Then by the rule for differentiating a composite function,

$$\frac{dy}{dx} = f'_u(u)\,\varphi'(x)$$

Hence,

$$dy = f'_u(u)\,\varphi'(x)\,dx$$

but $\varphi'(x)\,dx = du$, therefore

$$dy = f'(u)\,du$$

Thus, *the differential of a composite function has the same form as it would have if the intermediate argument were the independent variable.* In other words, *the form of the differential does not depend on whether the argument of the function is an independent variable or the function of another argument.* This important property of a differential, called *the preservation of the form of the differential*, will be widely used later on.

Example 8. Given a function $y=\sin\sqrt{x}$. Find dy.

Solution. Representing the given function as a composite one:

$$y=\sin u, \quad u=\sqrt{x}$$

we find

$$dy=\cos u\,\frac{1}{2\sqrt{x}}\,dx$$

but $\frac{1}{2\sqrt{x}}\,dx=du$, so we can write

$$dy=\cos u\,du$$

or

$$dy=\cos(\sqrt{x})\,d(\sqrt{x}).$$

THE GEOMETRIC MEANING OF THE DIFFERENTIAL

Let us consider the function

$$y=f(x)$$

and the curve it represents (Fig. 86).

On the curve $y=f(x)$, take an arbitrary point $M(x, y)$, draw a line tangent to the curve at this point and denote by α the angle* which the tangent line forms with the positive x-axis. Increase the independent variable by Δx; then the function will change by $\Delta y=NM_1$. To the values $x+\Delta x$, $y+\Delta y$ on the curve $y=f(x)$ there will correspond the point $M_1(x+\Delta x, y+\Delta y)$.

From the triangle MNT we find

$$NT=MN\tan\alpha$$

Since

$$\tan\alpha=f'(x), \quad MN=\Delta x$$

we get

$$NT=f'(x)\,\Delta x$$

But by the definition of a differential $f'(x)\,\Delta x=dy$. Thus,

$$NT=dy$$

The equation signifies that *the differential of a function $f(x)$, which corresponds to the given values x and Δx, is equal to the increment in the ordinate of the line tangent to the curve $y=f(x)$ at the given point x.*

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* Assuming that the function $f(x)$ has a finite derivative at the point x, we get $\alpha\neq\frac{\pi}{2}$.

From Fig. 86 it follows directly that

$$M_1T = \Delta y - dy$$

By what has already been proved, $\frac{M_1T}{NT} \to 0$ as $\Delta x \to 0$.

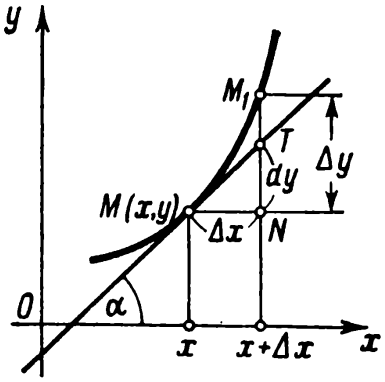

Fig. 86

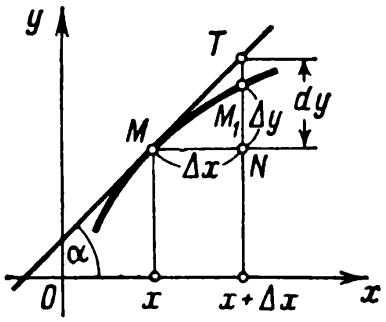

Fig. 87

One should not think that the increment Δy is always greater than dy. For instance, in Fig. 87,

$$\Delta y = M_1N, \quad dy = NT, \quad \text{and} \quad \Delta y < dy$$