AN IMPLICIT FUNCTION AND ITS DIFFERENTIATION

Let the values of two variables x and y be related by some equation, which we can symbolize as follows:

$$F(x, y)=0 \tag{1}$$

If the function $y=f(x)$, defined on some interval (a, b), is such that equation (1) becomes an identity in x when the expression $f(x)$ is substituted into it in place of y, the function $y=f(x)$ is an *implicit function* defined by equation (1).

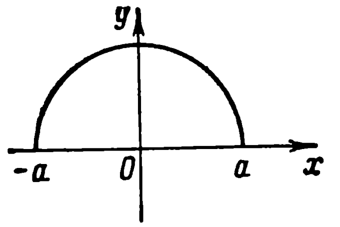

Fig. 64

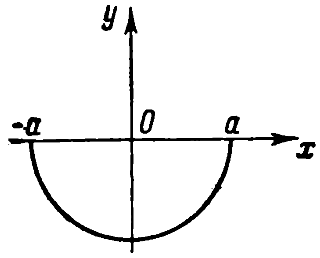

Fig. 65

For example, the equation

$$x^2+y^2-a^2=0 \tag{2}$$

defines implicitly the following elementary functions (Figs. 64 and 65):

$$y=\sqrt{a^2-x^2} \tag{3}$$

$$y=-\sqrt{a^2-x^2} \tag{4}$$

Indeed, substitution into equation (2) yields the identity

$$x^2+(a^2-x^2)-a^2=0$$

Expressions (3) and (4) were obtained by solving equation (2) for y. But not every implicitly defined function may be represented explicitly, that is, in the form $y=f(x)$, * where $f(x)$ is an elementary function.

For instance, functions defined by the equations

$$y^6-y-x^2=0$$

or

$$y-x-\frac{1}{4}\sin y=0$$

are not expressible in terms of elementary functions; that is, these equations cannot be solved for y.

* If a function is defined by an equation of the form $y=f(x)$, one says that the function is *defined explicitly* or is *explicit*.

Note 1. Observe that the terms "explicit function" and "implicit function" do not characterize the nature of the function but merely the way it is defined. Every explicit function $y=f(x)$ may also be represented as an implicit function $y-f(x)=0$.

We shall now give the rule for finding the derivative of an implicit function without transforming it into an explicit one, that is, without representing it in the form $y=f(x)$.

Assume the function is defined by the equation

$$x^2+y^2-a^2=0$$

Here, if y is a function of x defined by this equation, then the equation is an identity.

Differentiating both sides of this identity with respect to x, and regarding y as a function of x, we get (via the rule for differentiating a composite function)

$$2x+2yy'=0$$

whence

$$y'=-\frac{x}{y}$$

Observe that if we were to differentiate the corresponding explicit function

$$y=\sqrt{a^2-x^2}$$

we would obtain

$$y'=-\frac{x}{\sqrt{a^2-x^2}}=-\frac{x}{y}$$

which is the same result.

Let us consider another case of an implicit function y of x:

$$y^6-y-x^2=0$$

Differentiate with respect to x:

$$6y^5y'-y'-2x=0$$

whence

$$y'=\frac{2x}{6y^5-1}$$

Note 2. From the foregoing examples it follows that to find the value of the derivative of an implicit function for a given value of the argument x, one also has to know the value of the function y for the given value of x.

DERIVATIVES OF A POWER FUNCTION FOR AN ARBITRARY REAL EXPONENT, OF A GENERAL EXPONENTIAL FUNCTION, AND OF A COMPOSITE EXPONENTIAL FUNCTION

Theorem 1. *The derivative of the function x^n, where n is any real number, is equal to nx^{n-1}; that is,*

$$\text{if } y = x^n, \text{ then } y' = nx^{n-1}. \tag{I'}$$

Proof. Let $x > 0$. Taking logarithms of this function, we get

$$\ln y = n \ln x$$

Differentiate, with respect to x, both sides of the equation obtained, taking y to be a function of x:

$$\frac{y'}{y} = n\frac{1}{x}, \quad y' = yn\frac{1}{x}$$

Substituting into this equation the value $y = x^n$, we finally get

$$y' = nx^{n-1}$$

It is easy to show that this formula holds true also for $x < 0$ provided x^n is meaningful. *

Theorem 2. *The derivative of the function a^x, where $a > 0$, is $a^x \ln a$; that is,*

$$\text{if } y = a^x, \text{ then } y' = a^x \ln a. \tag{XIV}$$

Proof. Taking logarithms of the equation $y = a^x$, we get

$$\ln y = x \ln a$$

Differentiate the equation obtained regarding y as a function of x:

$$\frac{1}{y}y' = \ln a, \quad y' = y \ln a$$

or

$$y' = a^x \ln a$$

If the base is $a = e$, then $\ln e = 1$ and we have the formula

$$y = e^x, \quad y' = e^x \tag{XIV'}$$

Example 1. Given the function

$$y = e^{x^2}$$

Represent it as a composite function by introducing the intermediate argument u:

$$y = e^u, \quad u = x^2$$

then

$$y'_u = e^u, \quad u'_x = 2x$$

---

* This formula was proved in Sec. 3.5, for the case when n is a *positive integer*. Formula (I) has now been proved for the general case (for any constant number n).

and, therefore,

$$y'_x = e^u \cdot 2x = e^{x^2} \cdot 2x$$

A composite exponential function is a function in which both the base and the exponent are functions of x, for instance, $(\sin x)^{x^2}$, $x^{\tan x}$, x^x, $(\ln x)^x$; generally, any function of the form

$$y = [u(x)]^{v(x)} \equiv u^v$$

is a composite exponential function.*

Theorem 3.

$$\textit{If}\ y = u^v,\ \textit{then}\ y' = vu^{v-1}u' + u^v v' \ln u. \qquad \text{(XV)}$$

Proof. Taking logarithms of the function y, we have

$$\ln y = v \ln u$$

Differentiating the resultant equation with respect to x, we get

$$\frac{1}{y} y' = v \frac{1}{u} u' + v' \ln u$$

whence

$$y' = y\left(v \frac{u'}{u} + v' \ln u\right)$$

Substituting into this equation the expression $y = u^v$, we obtain

$$y' = vu^{v-1}u' + u^v v' \ln u$$

Thus, the derivative of a composite exponential function consists of two terms: the first term is obtained by assuming, when differentiating, that u is a function of x and v is a **constant** (that is to say, if we regard u^v as a **power** function); the second term is obtained on the assumption that v is a function of x, and $u = \text{const}$ (i. e., if we regard u^v as an **exponential** function).

Example 2. If $y = x^x$, then $y' = xx^{x-1}(x') + x^x(x') \ln x$ or

$$y' = x^x + x^x \ln x = x^x(1 + \ln x)$$

Example 3. If $y = (\sin x)^{x^2}$, then

$$\begin{aligned} y' &= x^2 (\sin x)^{x^2-1} (\sin x)' + (\sin x)^{x^2} (x^2)' \ln \sin x \\ &= x^2 (\sin x)^{x^2-1} \cos x + (\sin x)^{x^2} 2x \ln \sin x \end{aligned}$$

The procedure applied in this section for finding derivatives (first finding the derivative **of the logarithm of the given function**) is widely used in differentiating functions. Very often the use of this method greatly simplifies calculations.

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* In the Soviet mathematical literature this function is also called an exponential-power function or a power-exponential function.

Example 4. Find the derivative of the function

$$y=\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}$$

Solution. Taking logarithms we get

$$\ln y=2\ln(x+1)+\frac{1}{2}\ln(x-1)-3\ln(x+4)-x$$

Differentiate both sides of this equation:

$$\frac{y'}{y}=\frac{2}{x+1}+\frac{1}{2(x-1)}-\frac{3}{x+4}-1$$

Multiplying by y and substituting, in place of y, the expression $\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}$, we get

$$y'=\frac{(x+1)^2\sqrt{x-1}}{(x+4)^3e^x}\left[\frac{2}{x+1}+\frac{1}{2(x-1)}-\frac{3}{x+4}-1\right]$$

Note. The expression $\frac{y'}{y}=(\ln y)'$, which is the derivative, with respect to x, of the natural logarithm of the given function $y=y(x)$, is called the *logarithmic derivative*.

AN INVERSE FUNCTION AND ITS DIFFERENTIATION

Take an increasing or decreasing function (Fig. 66)

$$y=f(x) \qquad (1)$$

defined in some interval (a, b) $(a<b)$ (see Sec. 1.6). Let $f(a)=c$, $f(b)=d$. For definiteness we shall henceforward consider an increasing function.

Let us consider two different values x_1 and x_2 in the interval (a, b). From the definition of an increasing function it follows that if $x_1<x_2$ and $y_1=f(x_1)$, $y_2=f(x_2)$, then $y_1<y_2$. Hence, to two different values x_1 and x_2 there correspond two different values of the function, y_1 and y_2. The converse is also true: if $y_1<y_2$, $y_1=f(x_1)$, and $y_2=f(x_2)$, then from the definition of an increasing function it follows that $x_1<x_2$. Thus, a one-to-one correspondence is established between the values of x and the corresponding values of y.

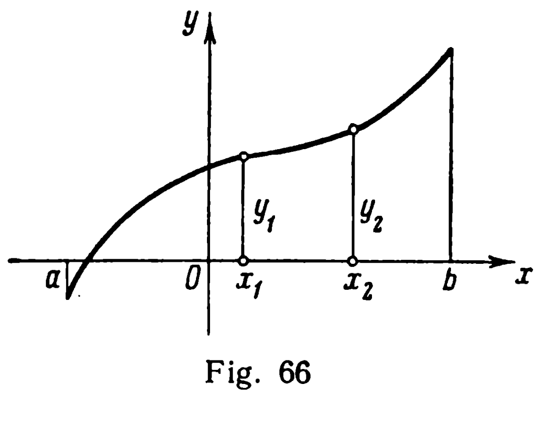

Fig. 66

Regarding these values of y as values of the argument and the values of x as values of the function, we get x as a function of y:

$$x=\varphi(y) \qquad (2)$$

This function is called the *inverse function* of $y=f(x)$. It is obvious too that the function $y=f(x)$ is the inverse of $x=\varphi(y)$. With similar reasoning it is possible to prove that a decreasing function also has an inverse.

Note 1. We state, without proof, that *if an increasing (or decreasing) function $y=f(x)$ is continuous on an interval $[a, b]$, where $f(a)=c$, $f(b)=d$, then the inverse function is defined and is continuous on the interval $[c, d]$.*

Example 1. Given the function $y=x^3$. This function is increasing on the infinite interval $-\infty < x < +\infty$; it has an inverse function $x=\sqrt[3]{y}$ (Fig. 67).

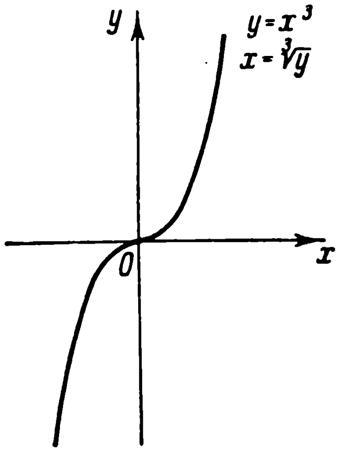

Fig. 67

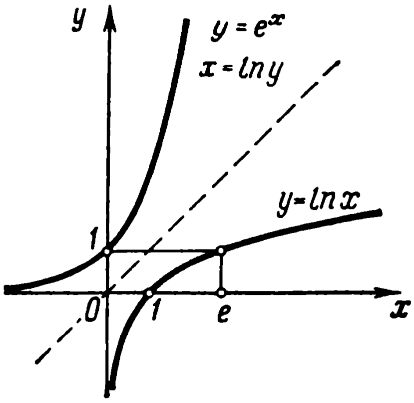

Fig. 68

It will be noted that the inverse function $x=\varphi(y)$ is found by solving the equation $y=f(x)$ for x.

Example 2. Given the function $y=e^x$. This function is increasing on the infinite interval $-\infty < x < +\infty$. It has an inverse $x=\ln y$. The domain of definition of the inverse function is $0 < y < +\infty$ (Fig. 68).

Note 2. If the function $y=f(x)$ is neither increasing nor decreasing on a certain interval, it can have several inverse functions.*

Example 3. The function $y=x^2$ is defined on an infinite interval $-\infty < x < +\infty$. It is neither increasing nor decreasing and does not have an inverse function. If we consider the interval $0 \leqslant x < +\infty$, then the function here is increasing and $x=\sqrt{y}$ is its inverse. But in the interval $-\infty < x < 0$ the function is decreasing and its inverse is $x=-\sqrt{y}$ (Fig. 69).

Note 3. If the functions $y=f(x)$ and $x=\varphi(y)$ are inverses of each other, their graphs are represented by a single curve. But if

---

* Let it be noted once again that when speaking of y as a function of x we have in mind that y is a single-valued function of x.

we again denote the argument of the inverse function by x, and the function by y, and then construct them in a single coordinate system, we will get two different graphs.

It will readily be seen that the graphs will be symmetric about the bisector of the first quadrantal angle.

Example 4. Fig. 68 gives the graphs of the function $y=e^x$ (or $x=\ln y$) and its inverse $y=\ln x$, which are considered in Example 2.

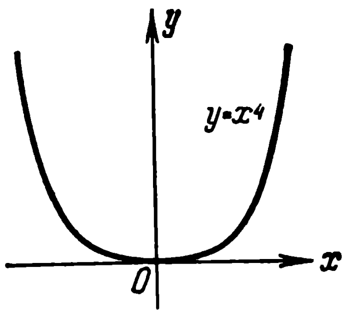

Fig. 69

Let us now prove a theorem that permits finding the derivative of a function $y=f(x)$ if we know the derivative of the inverse function.

Theorem. *If for the function*

$$y=f(x) \tag{1}$$

there exists an inverse function

$$x=\varphi(y) \tag{2}$$

which at the point under consideration y has a nonzero derivative $\varphi'(y)$, then at the corresponding point x the function $y=f(x)$ has a derivative $f'(x)$ equal to $\frac{1}{\varphi'(y)}$; that is, the following formula is true:

$$f'(x)=\frac{1}{\varphi'(y)} \tag{XVI}$$

Thus, the derivative of one of two inverse functions is equal to unity divided by the derivative of the second function for corresponding values of x and y.*

Proof. Take the increment Δy. Then, by (2), we have

$$\Delta x=\varphi(y+\Delta y)-\varphi(y)$$

Since $\varphi(y)$ is a monotonic function, it follows that $\Delta x\neq 0$. We write the identity

$$\frac{\Delta y}{\Delta x}=\frac{1}{\frac{\Delta x}{\Delta y}} \tag{3}$$

* When we write $f'(x)$ or y'_x we regard x as the independent variable when evaluating the derivative; but when we write $\varphi'(y)$ or x'_y we assume that y is the independent variable when evaluating the derivative. It should be noted that **after differentiating with respect to y, as indicated on the right side of formula (XVI), $f(x)$ must be substituted for y.**

Since the function $\varphi(y)$ is continuous, then $\Delta x \to 0$ as $\Delta y \to 0$. Passing to the limit as $\Delta y \to 0$ in both members of (3), we get

$$y'_x = \frac{1}{x'_y} \text{ or } f'(x) = \frac{1}{\varphi'(y)}$$

In other words, we obtain formula XVI.

Note. If one takes advantage of the theorem on differentiating a composite function, then formula XVI may be obtained in the following manner.

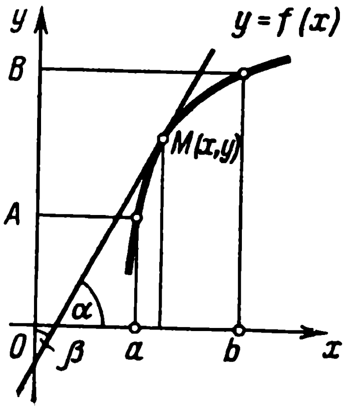

Fig. 70

Differentiate both members of (2) with respect to x, taking y to be a function of x. This yields $1 = \varphi'(y)\, y'_x$, whence

$$y'_x = \frac{1}{\varphi'(y)}$$

The result obtained is clearly illustrated geometrically. Consider the graph of the function $y = f(x)$ (Fig. 70). This curve will also be the graph of the function $x = \varphi(y)$, where x is now regarded as the function and y as the independent variable. Take some point $M(x, y)$ on this curve. Draw a tangent to the curve at this point. Denote by α and β the angles formed by the given tangent and the positive x- and y-axes. On the basis of the results of Sec. 3.3 concerning the geometrical meaning of a derivative we have

$$\left.\begin{aligned} f'(x) &= \tan\alpha \\ \varphi'(y) &= \tan\beta \end{aligned}\right\} \tag{4}$$

From Fig. 70 it follows directly that if $\alpha < \frac{\pi}{2}$, then $\beta = \frac{\pi}{2} - \alpha$. But if $\alpha > \frac{\pi}{2}$, then, as is readily seen, $\beta = \frac{3\pi}{2} - \alpha$. Hence, in any case $\tan\beta = \cot\alpha$, whence $\tan\alpha\tan\beta = \tan\alpha\cot\alpha = 1$, or $\tan\alpha = \frac{1}{\tan\beta}$. Substituting the expressions for $\tan\alpha$ and $\tan\beta$ from formula (4), we get

$$f'(x) = \frac{1}{\varphi'(y)}$$

INVERSE TRIGONOMETRIC FUNCTIONS AND THEIR DIFFERENTIATION

(1) The function $y = \arcsin x$.

Let us consider the function

$$x = \sin y \tag{1}$$

and construct its graph by directing the y-axis vertically upwards (Fig. 71). This function is defined in the infinite interval $-\infty < y < +\infty$. Over the interval $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, the function $x = \sin y$ is increasing and its values fill the interval $-1 \leqslant x \leqslant 1$. For this reason, the function $x = \sin y$ has an inverse which is denoted by

$$y = \arcsin x\,^*$$

This function is defined on the interval $-1 \leqslant x \leqslant 1$, and its values fill the interval $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$. In Fig. 71, the graph of $y = \arcsin x$ is shown by the heavy line.

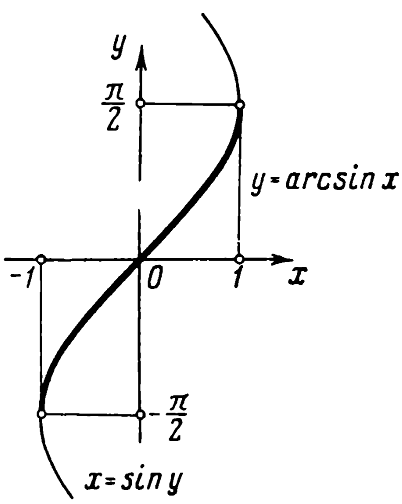

Fig. 71

Theorem 1. *The derivative of the function* $\arcsin x$ *is equal to* $\frac{1}{\sqrt{1-x^2}}$; *i. e.,*

$$\text{if } y = \arcsin x, \text{ then } y' = \frac{1}{\sqrt{1-x^2}}. \qquad \text{(XVII)}$$

Proof. On the basis of (1) we have

$$x'_y = \cos y$$

By the rule for differentiating an inverse function,

$$y'_x = \frac{1}{x'_y} = \frac{1}{\cos y}$$

but

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$$

therefore,

$$y'_x = \frac{1}{\sqrt{1-x^2}}$$

The sign in front of the radical is plus because the function $y = \arcsin x$ takes on values in the interval $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, and, consequently, $\cos y \geqslant 0$.

Example 1. $y = \arcsin e^x$,

$$y' = \frac{1}{\sqrt{1-(e^x)^2}}(e^x)' = \frac{e^x}{\sqrt{1-e^{2x}}}$$

---

* It may be noted that the familiar equation $y = \operatorname{Arcsin} x$ of trigonometry is another way of writing (1). Here (for a given x) y denotes the set of values of angles whose sine is equal to x.

Example 2. $y=\left(\arcsin\frac{1}{x}\right)^2$.

$$y'=2\arcsin\frac{1}{x}\frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(\frac{1}{x}\right)'=-2\arcsin\frac{1}{x}\frac{1}{x\sqrt{x^2-1}}$$

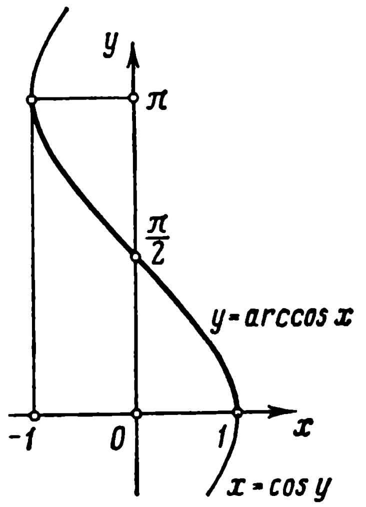

Fig. 72

(2) The function $y=\arccos x$.

As before, we consider the function

$$x=\cos y \qquad (2)$$

and construct its graph with the y-axis extending upwards (Fig. 72). This function is defined on the infinite interval $-\infty<y<+\infty$. On the interval $0\leqslant y\leqslant\pi$, the function $x=\cos y$ is decreasing and has an inverse that we denote

$$y=\arccos x$$

This function is defined on the interval $-1\leqslant x\leqslant 1$. The values of the function fill the interval $\pi\geqslant y\geqslant 0$. In Fig. 72, the function $y=\arccos x$ is depicted by the heavy line.

Theorem 2. *The derivative of the function* $\arccos x$ *is* $-\frac{1}{\sqrt{1-x^2}}$; *i. e.,*

$$\text{if } y=\arccos x, \text{ then } y'=-\frac{1}{\sqrt{1-x^2}}. \qquad \text{(XVIII)}$$

Proof. From (2) we have

$$x'_y=-\sin y$$

Hence

$$y'_x=\frac{1}{x'_y}=-\frac{1}{\sin y}=-\frac{1}{\sqrt{1-\cos^2 y}}$$

But $\cos y=x$, and so

$$y'_x=-\frac{1}{\sqrt{1-x^2}}$$

In $\sin y=\sqrt{1-\cos^2 y}$ the radical is taken with the plus sign, since the function $y=\arccos x$ is defined on the interval $0\leqslant y\leqslant\pi$ and, consequently, $\sin y\geqslant 0$.

Example 3. $y=\arccos(\tan x)$,

$$y'=-\frac{1}{\sqrt{1-\tan^2 x}}(\tan x)'=-\frac{1}{\sqrt{1-\tan^2 x}}\frac{1}{\cos^2 x}$$

(3) The function $y = \arctan x$.
We consider the function

$$x = \tan y \qquad (3)$$

and construct its graph (Fig. 73). This function is defined for all values of y except $y = (2k+1)\frac{\pi}{2}$ ($k = 0, \pm 1, \pm 2, \ldots$). On the interval $-\frac{\pi}{2} < y < \frac{\pi}{2}$ the function $x = \tan y$ is increasing and has an inverse:

$$y = \arctan x$$

This function is defined on the interval $-\infty < x < +\infty$. The values of the function fill the interval $-\frac{\pi}{2} < y < \frac{\pi}{2}$. In Fig. 73, the graph of the function $y = \arctan x$ is shown as a heavy line.

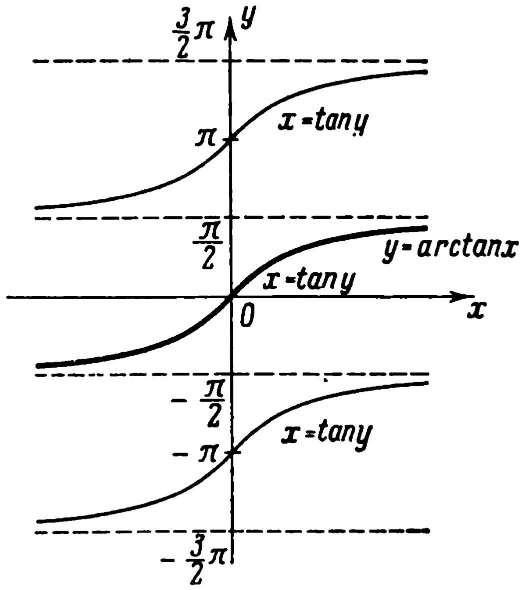

Fig. 73

Theorem 3. *The derivative of the function* $\arctan x$ *is* $\frac{1}{1+x^2}$; *i.e.*,

$$\textit{if } y = \arctan x, \quad \textit{then } y' = \frac{1}{1+x^2}. \qquad \text{(XIX)}$$

Proof. From (3) we have

$$x'_y = \frac{1}{\cos^2 y}$$

Hence

$$y'_x = \frac{1}{x'_y} = \cos^2 y$$

but

$$\cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y}$$

since $\tan y = x$, we get, finally,

$$y' = \frac{1}{1+x^2}$$

Example 4. $y = (\arctan x)^4$,

$$y' = 4(\arctan x)^3 (\arctan x)' = 4(\arctan x)^3 \frac{1}{1+x^2}$$

(4) The function $y = \operatorname{arccot} x$.

Consider the function

$$x = \cot y \tag{4}$$

This function is defined for all values of y except $y = k\pi$ ($k = 0$, ± 1, ± 2, ...). The graph of this function is shown in Fig. 74.

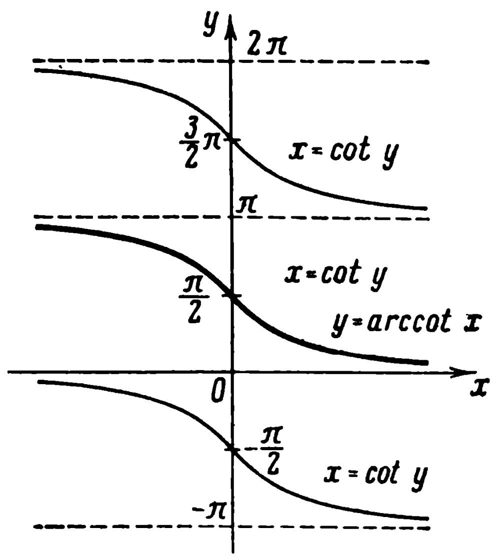

Fig. 74

On the interval $0 < y < \pi$, the function $x = \cot y$ is decreasing and has an inverse:

$$y = \operatorname{arccot} y$$

Consequently, this function is defined on the infinite interval $-\infty < x < +\infty$, and its values fill the interval $\pi > y > 0$.

Theorem 4. *The derivative of the function* $\operatorname{arccot} x$ *is* $-\frac{1}{1+x^2}$; *i.e.*,

$$\textit{if } y = \operatorname{arccot} x, \textit{ then } y' = -\frac{1}{1+x^2}. \tag{XX}$$

Proof. From (4) we have

$$x'_y = -\frac{1}{\sin^2 y}$$

Hence

$$y'_x = -\sin^2 y = -\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y}$$

But

$$\cot y = x$$

Therefore

$$y'_x = -\frac{1}{1+x^2}$$

BASIC DIFFERENTIATION FORMULAS

Let us now bring together into a single table all the basic formulas and rules of differentiation derived in the preceding sections.

$$y = \text{const}, \quad y' = 0$$

Power function:

$$y = x^\alpha, \quad y' = \alpha x^{\alpha-1}$$

particular instances:

$$y = \sqrt{x}, \quad y' = \frac{1}{2\sqrt{x}}$$

$$y = \frac{1}{x}, \quad y' = -\frac{1}{x^2}$$

Trigonometric functions:

$$y = \sin x, \quad y' = \cos x$$
$$y = \cos x, \quad y' = -\sin x$$
$$y = \tan x, \quad y' = \frac{1}{\cos^2 x}$$
$$y = \cot x, \quad y' = -\frac{1}{\sin^2 x}$$

Inverse trigonometric functions:

$$y = \arcsin x, \quad y' = \frac{1}{\sqrt{1-x^2}}$$
$$y = \arccos x, \quad y' = -\frac{1}{\sqrt{1-x^2}}$$
$$y = \arctan x, \quad y' = \frac{1}{1+x^2}$$
$$y = \operatorname{arccot} x, \quad y' = -\frac{1}{1+x^2}$$

Exponential function:

$$y = a^x, \quad y' = a^x \ln a$$

in particular,

$$y = e^x, \quad y' = e^x$$

Logarithmic function:

$$y = \log_a x, \quad y' = \frac{1}{x} \log_a e$$

in particular,

$$y = \ln x, \quad y' = \frac{1}{x}$$

General rules for differentiation:

$$y = Cu(x), \qquad y' = Cu'(x) \quad (C = \text{const})$$
$$y = u + v - w, \quad y' = u' + v' - w'$$
$$y = uv, \qquad y' = u'v + uv'$$
$$y = \frac{u}{v}, \qquad y' = \frac{u'v - uv'}{v^2}$$
$$\left.\begin{aligned} y &= f(u), \\ u &= \varphi(x), \end{aligned}\right\} \qquad y'_x = f'_u(u)\,\varphi'_x(x)$$
$$y = u^v, \qquad y' = vu^{v-1}u' + u^v v' \ln u$$

If $y = f(x)$, $x = \varphi(y)$, where f and φ are inverse functions, then

$$f'(x) = \frac{1}{\varphi'(y)}, \quad \text{where } y = f(x)$$