Lecture 7

VELOCITY OF MOTION

Let us consider the rectilinear motion of some solid, say a stone, thrown vertically upwards, or the motion of a piston in the cylinder of an engine, etc. Idealizing the situation and disregarding dimensions and shapes, we shall always represent such a body in the form of a moving point M. The distance s of the moving point reckoned from some initial position M_0 will depend on the time t; in other words, s will be a function of time t:

$$s = f(t) \tag{1}$$

At some instant of time* t, let the moving point M be at a distance s from the initial position M_0, and at some later instant $t+\Delta t$ let the point be at M_1, a distance $s+\Delta s$ from the initial position (Fig. 57). Thus, during the interval of time Δt the distance s changed by the quantity Δs. In such cases, one says that during the time Δt the quantity s received an increment Δs.

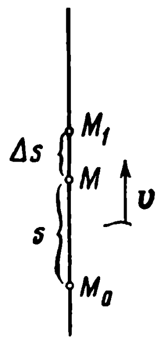

Fig. 57

Let us consider the ratio $\frac{\Delta s}{\Delta t}$; it gives us the average velocity of motion of the point during the time Δt:

$$v_{av} = \frac{\Delta s}{\Delta t} \tag{2}$$

The average velocity cannot in all cases give an exact picture of the rate of translation of the point M at time t. If, for example, the body moved very fast at the beginning of the interval Δt and very slow at the end, the average velocity obviously cannot reflect these peculiarities in the motion of the point and give us a correct idea of the true velocity of motion at time t. In order to express more precisely this true velocity in terms of the average velocity, one has to take a smaller interval of time Δt. The most complete description of the rate of motion of the point at time t is given

* Here and henceforward we shall denote the specific value of a variable and the variable itself by the same letter.

by the limit which the average velocity approaches as $\Delta t \longrightarrow 0$. This limit is called the **rate of motion at a given instant:**

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \tag{3}$$

Thus, the *rate (velocity) of motion at a given instant* is the limit of the ratio of increment in path Δs to increment in time Δt, as the time increment approaches zero.

Let us write equation (3) in full. Since

$$\Delta s = f(t+\Delta t) - f(t),$$

it follows that

$$v = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{3'}$$

This is the velocity of nonuniform motion. It is thus obvious that the notion of velocity of nonuniform motion is intimately related to the concept of a limit. It is only with the aid of the limit concept that we can determine the velocity of nonuniform motion.

From formula (3′) it follows that v is independent of the increment in time Δt, but depends on the value of t and the type of function $f(t)$.

Example. Find the velocity of uniformly accelerated motion at an arbitrary time t and at $t = 2$ sec if the relation of the path traversed to the time is expressed by the formula

$$s = \frac{1}{2} g t^2$$

Solution. At time t we have $s = \frac{1}{2} g t^2$; at time $t + \Delta t$ we get

$$s + \Delta s = \frac{1}{2} g (t + \Delta t)^2 = \frac{1}{2} g (t^2 + 2t\,\Delta t + \Delta t^2)$$

We find Δs:

$$\Delta s = \frac{1}{2} g (t^2 + 2t\,\Delta t + \Delta t^2) - \frac{1}{2} g t^2 = g t\,\Delta t + \frac{1}{2} g\,\Delta t^2$$

We form the ratio $\frac{\Delta s}{\Delta t}$:

$$\frac{\Delta s}{\Delta t} = \frac{g t\,\Delta t + \frac{1}{2} g\,\Delta t^2}{\Delta t} = g t + \frac{1}{2} g\,\Delta t$$

By definition we have

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \left(g t + \frac{1}{2} g\,\Delta t \right) = g t$$

Thus, the velocity at an arbitrary time t is $v = g t$.
At $t = 2$ we have $(v)_{t=2} = g \cdot 2 = 9.8 \cdot 2 = 19.6$ m/sec.

3.2 THE DEFINITION OF A DERIVATIVE

Let there be a function

$$y = f(x) \tag{1}$$

defined in a certain interval. The function $y = f(x)$ has a definite value for each value of the argument x in this interval.

Let the argument x receive a certain increment Δx (it is immaterial whether it is positive or negative). Then the function y will receive a certain increment Δy. Thus, for the value of the argument x we will have $y = f(x)$, for the value of the argument $x + \Delta x$ we will have $y + \Delta y = f(x + \Delta x)$.

Let us find the increment of the function Δy:

$$\Delta y = f(x + \Delta x) - f(x) \tag{2}$$

Forming the ratio of the increment of the function to the increment of the argument, we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{3}$$

We then find the limit of this ratio as $\Delta x \to 0$. If this limit exists, it is called the **derivative** of the given function $f(x)$ and is denoted $f'(x)$. Thus, by definition,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

or

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{4}$$

Consequently, the *derivative* of a given function $y = f(x)$ with respect to the argument x is the limit of the ratio of the increment in the function Δy to the increment in the argument Δx, when the latter approaches zero in arbitrary fashion.

It will be noted that in the general case, the derivative $f'(x)$ has a definite value for each value of x, which means that the derivative is also a **function** of x.

The designation $f'(x)$ is not the only one used for a derivative. Alternative symbols are

$$y', \; y'_x, \; \frac{dy}{dx}$$

The specific value of the derivative for $x = a$ is denoted $f'(a)$ or $y'|_{x=a}$.

The operation of finding the derivative of a function $f(x)$ is called *differentiation* of the function.

Example 1. Given the function $y=x^2$, find its derivative y':
(1) at an arbitrary point x,
(2) at $x=3$.

Solution. (1) For the value of the argument x, we have $y=x^2$. When the value of the argument is $x+\Delta x$, we have $y+\Delta y=(x+\Delta x)^2$.

Find the increment of the function:

$$\Delta y=(x+\Delta x)^2-x^2=2x\Delta x+(\Delta x)^2$$

Forming the ratio $\frac{\Delta y}{\Delta x}$, we have

$$\frac{\Delta y}{\Delta x}=\frac{2x\Delta x+(\Delta x)^2}{\Delta x}=2x+\Delta x$$

Passing to the limit, we get the derivative of the given function:

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}(2x+\Delta x)=2x$$

Hence, the derivative of the function $y=x^2$ at an arbitrary point is $y'=2x$.

(2) When $x=3$ we have

$$y'|_{x=3}=2\cdot 3=6$$

Example 2. $y=\frac{1}{x}$; find y'.

Solution. Reasoning as before, we get

$$y=\frac{1}{x},\quad y+\Delta y=\frac{1}{x+\Delta x},$$

$$\Delta y=\frac{1}{x+\Delta x}-\frac{1}{x}=\frac{x-x-\Delta x}{x(x+\Delta x)}=-\frac{\Delta x}{x(x+\Delta x)},$$

$$\frac{\Delta y}{\Delta x}=-\frac{1}{x(x+\Delta x)},$$

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left[-\frac{1}{x(x+\Delta x)}\right]=-\frac{1}{x^2}$$

Note. In the preceding section it was established that if the dependence upon time t of the distance s of a moving point is expressed by the formula

$$s=f(t)$$

the velocity v at time t is expressed by the formula

$$v=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}=\lim_{\Delta t\to 0}\frac{f(t+\Delta t)-f(t)}{\Delta t}$$

Hence

$$v=s'_t=f'(t)$$

or, the velocity is equal to the derivative* of the distance with respect to the time.

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* When we say "the derivative with respect to x" or "the derivative with respect to t" we mean that in computing the derivative we consider the variable x (or the time t, etc.) the argument (independent variable).

3.3 GEOMETRIC MEANING OF THE DERIVATIVE

We approached the notion of a derivative by regarding the velocity of a moving body (point), that is to say, by proceeding from **mechanical** concepts. We shall now give a no less important **geometric** interpretation of the derivative. To do this we must first define a **tangent line** to a curve at a given point.

We take a curve with a fixed point M_0 on it. Taking a point M_1 on the curve we draw the secant M_0M_1 (Fig. 58). If the point M_1 approaches the point M_0 without limit, the secant M_0M_1 will occupy various positions M_0M_1', M_0M_1'', and so on.

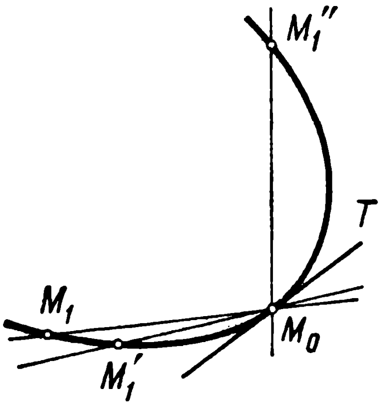

Fig. 58

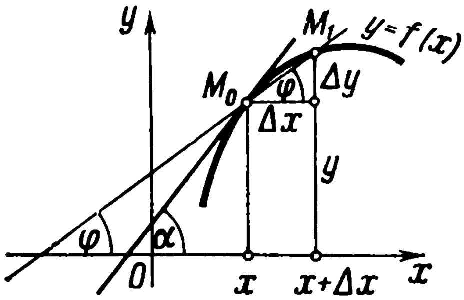

Fig. 59

If, in the unbounded approach of the point M_1 (along the curve) to the point M_0 **from either side**, the secant tends to occupy the position of a definite straight line M_0T, this line is called the *tangent* to the curve at the point M_0 (the concept "tends to occupy" will be explained later on).

Let us consider the function $f(x)$ and the corresponding curve

$$y = f(x)$$

in a rectangular coordinate system (Fig. 59). At a certain value of x the function has the value $y = f(x)$. Corresponding to these values of x and y on the curve we have the point $M_0(x, y)$. Let us increase the argument x by Δx. Corresponding to the new value of the argument, $x+\Delta x$, we have an increased value of the function, $y+\Delta y = f(x+\Delta x)$. The corresponding point on the curve will be $M_1(x+\Delta x,\ y+\Delta y)$. Draw the secant M_0M_1 and denote by φ the angle formed by the secant and the positive x-axis. Form the ratio $\frac{\Delta y}{\Delta x}$. From Fig. 59 it follows immediately that

$$\frac{\Delta y}{\Delta x} = \tan\varphi \qquad (1)$$

Now if Δx approaches zero, the point M_1 will move along the curve always approaching M_0. The secant M_0M_1 will turn about M_0 and the angle φ will change with Δx. If as $\Delta x \to 0$ the angle φ approaches a certain limit α, the straight line passing through M_0 and forming an angle α with the positive x-axis will be the sought-for tangent line. It is easy to find its slope:

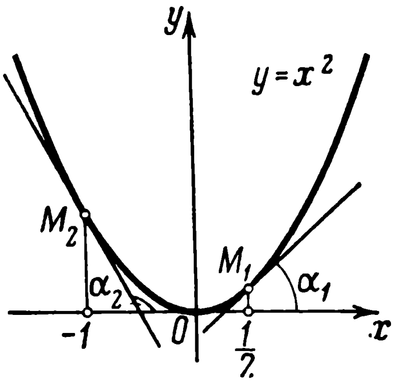

Fig. 60

$$\tan \alpha = \lim_{\Delta x \to 0} \tan \varphi = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$$

Hence,

$$f'(x) = \tan \alpha \tag{2}$$

which means that *the value of the derivative $f'(x)$, for a given value of the argument x, is equal to the tangent of the angle formed with the positive x-axis by the line tangent to the graph of the function $f(x)$ at the corresponding point $M_0(x, y)$.*

Example. Find the tangents of the angles of inclination of the tangent line to the curve $y = x^2$ at the points $M_1\left(\frac{1}{2}, \frac{1}{4}\right)$, $M_2(-1, 1)$ (Fig. 60).

Solution. On the basis of Example 1, Sec. 3.2, we have $y' = 2x$; hence,

$$\tan \alpha_1 = y' \Big|_{x=\frac{1}{2}} = 1, \quad \tan \alpha_2 = y' \Big|_{x=-1} = -2$$

DIFFERENTIABILITY OF FUNCTIONS

Definition. If the function

$$y = f(x) \tag{1}$$

has a derivative at the point $x = x_0$, that is, if there exists

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{2}$$

we say that for the given value $x = x_0$ the function is *differentiable* or (which is the same thing) has a derivative.

If a function is differentiable **at every point** of some interval $[a, b]$ or (a, b), we say that it is *differentiable over the interval.*

Theorem. *If a function $y = f(x)$ is differentiable at some point $x = x_0$, it is continuous at that point.*

Indeed, if

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)$$

then

$$\frac{\Delta y}{\Delta x}=f'(x_0)+\gamma \ .$$

where γ is a quantity that approaches zero as $\Delta x \to 0$. But then

$$\Delta y = f'(x_0)\,\Delta x + \gamma \Delta x$$

whence it follows that $\Delta y \to 0$ as $\Delta x \to 0$; and this means that the function $f(x)$ is continuous at the point x_0 (see Sec. 2.9).

In other words, **a function cannot have a derivative at points of discontinuity**. The converse is not true; from the fact that at some point $x=x_0$ the function $y=f(x)$ is continuous, it does not yet follow that it is differentiable at that point: the function $f(x)$ may not have a derivative at the point x_0. To convince ourselves of this, let us examine several cases.

Example 1. A function $f(x)$ is defined on an interval [0, 2] as follows (see Fig. 61):

$$\begin{aligned} f(x)&=x && \text{when } 0 \leqslant x \leqslant 1,\\ f(x)&=2x-1 && \text{when } 1 < x \leqslant 2 \end{aligned}$$

At $x=1$ the function has no derivative, although it is continuous at this point.

Indeed, when $\Delta x > 0$ we have

$$\lim_{\Delta x \to 0} \frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[2(1+\Delta x)-1]-[2\cdot 1-1]}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = 2$$

when $\Delta x < 0$ we get

$$\lim_{\Delta x \to 0} \frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(1+\Delta x)-1}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

Thus, this limit depends on the sign of Δx, and this means that the function has no derivative* at the point $x=1$. Geometrically, this is in accord with the fact that at the point $x=1$ the given "curve" does not have a definite tangent line.

Now the continuity of the function at the point $x=1$ follows from the fact that

$$\begin{aligned} \Delta y&=\Delta x && \text{when } \Delta x < 0,\\ \Delta y&=2\Delta x && \text{when } \Delta x > 0 \end{aligned}$$

and, therefore, in both cases $\Delta y \to 0$ as $\Delta x \to 0$.

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* The definition of a derivative requires that the ratio $\frac{\Delta y}{\Delta x}$ should (as $\Delta x \to 0$) approach one and the same limit regardless of *the way* in which Δx approaches zero.

Example 2. A function $y=\sqrt[3]{x}$, the graph of which is shown in Fig. 62, is defined and continuous for all values of the independent variable.

Let us try to find out whether this function has a derivative at $x=0$; to do this, we find the values of the function at $x=0$ and at $x=0+\Delta x$: at $x=0$ we have $y=0$. at $x=0+\Delta x$ we have $y+\Delta y=\sqrt[3]{\Delta x}$.

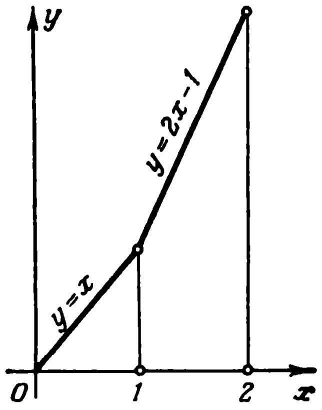

Fig. 61

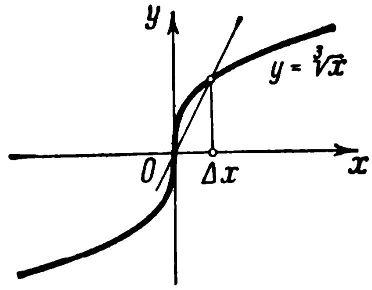

Fig. 62

Therefore,

$$\Delta y=\sqrt[3]{\Delta x}$$

Find the limit of the ratio of the increment of the function to the increment of the argument:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\sqrt[3]{\Delta x}}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{\Delta x^2}}=+\infty$$

Thus, the ratio of the increment of the function to the increment of the argument at the point $x=0$ approaches infinity as $\Delta x\to 0$ (hence there is no limit). Consequently, this function is not differentiable at the point $x=0$. The tangent to the curve at this point forms, with the x-axis, an angle $\frac{\pi}{2}$, which means that it coincides with the y-axis.

THE DERIVATIVE OF THE FUNCTION $y=x^n$, n A POSITIVE INTEGER

To find the derivative of a given function $y=f(x)$, it is necessary to carry out the following operations (on the basis of the general definition of a derivative):

(1) increase the argument x by Δx, calculate the increased value of the function:

$$y+\Delta y=f(x+\Delta x)$$

(2) find the corresponding increment in the function:

$$\Delta y=f(x+\Delta x)-f(x)$$

(3) form the ratio of the increment in the function to the increment in the argument:

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

(4) find the limit of this ratio as $\Delta x \to 0$:

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Here and in the following sections, we shall apply this general method for evaluating the derivatives of certain elementary functions.

Theorem. *The derivative of the function $y=x^n$, where n is a positive integer, is equal to nx^{n-1}; that is,*

$$\textit{if } y=x^n, \textit{ then } y'=nx^{n-1} \tag{I}$$

Proof. We have the function

$$y=x^n$$

(1) If x receives an increment Δx, then

$$y+\Delta y=(x+\Delta x)^n$$

(2) Applying Newton's binomial theorem, we get

$$\Delta y=(x+\Delta x)^n-x^n=$$
$$=x^n+\frac{n}{1}x^{n-1}\Delta x+\frac{n(n-1)}{1\cdot 2}x^{n-2}(\Delta x)^2+\ldots+(\Delta x)^n-x^n$$

or

$$\Delta y=nx^{n-1}\Delta x+\frac{n(n-1)}{1\cdot 2}x^{n-2}(\Delta x)^2+\ldots+(\Delta x)^n$$

(3) We find the ratio

$$\frac{\Delta y}{\Delta x}=nx^{n-1}+\frac{n(n-1)}{1\cdot 2}x^{n-2}\Delta x+\ldots+(\Delta x)^{n-1}$$

(4) Then we find the limit of this ratio:

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$
$$=\lim_{\Delta x\to 0}\left[nx^{n-1}+\frac{n(n-1)}{1\cdot 2}x^{n-2}\Delta x+\ldots+(\Delta x)^{n-1}\right]=nx^{n-1}$$

consequently, $y'=nx^{n-1}$, and the theorem is proved.

Example 1. $y=x^5$, $y'=5x^{5-1}=5x^4$.

Example 2. $y=x$, $y'=1x^{1-1}$, $y'=1$. The latter result has a simple geometric interpretation: the tangent to the straight line $y=x$ for any value of

x coincides with this line and, consequently, forms with the positive x-axis an angle, whose tangent is 1.

Note that formula (I) also holds true when n is fractional or negative. (This will be proved in Sec. 3.12).

Example 3. $y=\sqrt{x}$.

Let us represent the function in the form of a power:

$$y=x^{\frac{1}{2}}$$

Then by formula (I), taking into consideration what we have just said, we get

$$y'=\frac{1}{2}x^{\frac{1}{2}-1}$$

or

$$y'=\frac{1}{2\sqrt{x}}$$

Example 4. $y=\frac{1}{x\sqrt{x}}$.

Represent y in the form of a power function:

$$y=x^{-\frac{3}{2}}$$

Then

$$y'=-\frac{3}{2}x^{-\frac{3}{2}-1}=-\frac{3}{2}x^{-\frac{5}{2}}=-\frac{3}{2x^2\sqrt{x}}$$

DERIVATIVES OF THE FUNCTIONS $y=\sin x$, $y=\cos x$

Theorem 1. *The derivative of* $\sin x$ *is* $\cos x$, *or*

$$\text{if } y=\sin x, \text{ then } y'=\cos x. \tag{II}$$

Proof. Increase the argument x by Δx; then

(1) $y+\Delta y=\sin(x+\Delta x)$;

(2) $\Delta y=\sin(x+\Delta x)-\sin x=2\sin\frac{x+\Delta x-x}{2}\cos\frac{x+\Delta x+x}{2}$

$$=2\sin\frac{\Delta x}{2}\cdot\cos\left(x+\frac{\Delta x}{2}\right);$$

(3) $\frac{\Delta y}{\Delta x}=\frac{2\sin\frac{\Delta x}{2}\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x}=\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cos\left(x+\frac{\Delta x}{2}\right)$;

(4) $y'=\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim\limits_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\lim\limits_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)$.

But since

$$\lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1$$

we get

$$y' = \lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) = \cos x$$

This latter equation is obtained on the grounds that $\cos x$ is a continuous function.

Theorem 2. *The derivative of* $\cos x$ *is* $-\sin x$, *or*

$$\text{if } y = \cos x, \text{ then } y' = -\sin x. \qquad \text{(III)}$$

Proof. Increase the argument x by Δx, then

$$y + \Delta y = \cos(x + \Delta x)$$

$$\Delta y = \cos(x + \Delta x) - \cos x = -2\sin\frac{x + \Delta x - x}{2}\sin\frac{x + \Delta x + x}{2}$$

$$= -2\sin\frac{\Delta x}{2}\sin\left(x + \frac{\Delta x}{2}\right)$$

$$\frac{\Delta y}{\Delta x} = -\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \sin\left(x + \frac{\Delta x}{2}\right)$$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} -\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} \sin\left(x + \frac{\Delta x}{2}\right) = -\lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right)$$

Taking into account the fact that $\sin x$ is a continuous function, we finally get

$$y' = -\sin x$$

DERIVATIVES OF: A CONSTANT, THE PRODUCT OF A CONSTANT BY A FUNCTION, A SUM, A PRODUCT, AND A QUOTIENT

Theorem 1. *The derivative of a constant is equal to zero; that is,*

$$\text{if } y = C, \text{ where } C = \text{const}, \text{ then } y' = 0. \qquad \text{(IV)}$$

Proof. $y = C$ is a function of x such that the values of it are equal to C for all x.

Hence, for any value of x,

$$y = f(x) = C$$

We increase the argument x by Δx ($\Delta x \neq 0$). Since the function y retains the value C for all values of the argument, we have

$$y + \Delta y = f(x + \Delta x) = C$$

Therefore, the increment of the function is

$$\Delta y = f(x + \Delta x) - f(x) = 0$$

the ratio of the increment of the function to the increment of the argument

$$\frac{\Delta y}{\Delta x} = 0$$

and, consequently,

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

that is,

$$y' = 0$$

The latter result has a simple geometric interpretation. The graph of the function $y = C$ is a straight line parallel to the x-axis. Obviously, the tangent to the graph at any one of its points coincides with this straight line and, therefore, forms with the x-axis an angle whose tangent y' is zero.

Theorem 2. *A constant factor may be taken outside the derivative sign, i.e.,*

$$\textit{if } y = Cu(x) \ (C = \text{const}), \textit{ then } y' = Cu'(x). \qquad \text{(V)}$$

Proof. Reasoning as in the proof of the preceding theorem, we have

$$y = Cu(x)$$
$$y + \Delta y = Cu(x + \Delta x)$$
$$\Delta y = Cu(x + \Delta x) - Cu(x) = C[u(x + \Delta x) - u(x)]$$
$$\frac{\Delta y}{\Delta x} = C\frac{u(x + \Delta x) - u(x)}{\Delta x}$$
$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = C \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}, \text{ i.e., } y' = Cu'(x)$$

Example 1. $y = 3\frac{1}{\sqrt{x}}$.

$$y' = 3\left(\frac{1}{\sqrt{x}}\right)' = 3\left(x^{-\frac{1}{2}}\right)' = 3\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} = -\frac{3}{2}x^{-\frac{3}{2}}$$

or

$$y' = -\frac{3}{2x\sqrt{x}}$$

Theorem 3. *The derivative of the sum of a finite number of differentiable functions is equal to the corresponding sum of the derivatives of these functions.* *

For the case of three terms, for example, we have

$$y = u(x) + v(x) + w(x), \quad y' = u'(x) + v'(x) + w'(x) \qquad \text{(VI)}$$

Proof. For the values of the argument x

$$y = u + v + w$$

(for the sake of brevity we drop the argument x in denoting the function).

For the value of the argument $x + \Delta x$ we have

$$y + \Delta y = (u + \Delta u) + (v + \Delta v) + (w + \Delta w)$$

where Δy, Δu, Δv, and Δw are increments of the functions y, u, v and w, which correspond to the increment Δx in the argument x. Hence,

$$\Delta y = \Delta u + \Delta v + \Delta w, \quad \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \frac{\Delta w}{\Delta x}$$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x}$$

or

$$y' = u'(x) + v'(x) + w'(x)$$

Example 2. $y = 3x^4 - \frac{1}{\sqrt[3]{x}}$.

$$y' = 3(x^4)' - \left(x^{-\frac{1}{3}}\right)' = 3 \cdot 4x^3 - \left(-\frac{1}{3}\right) x^{-\frac{1}{3}-1}$$

and so

$$y' = 12x^3 + \frac{1}{3} \frac{1}{x\sqrt[3]{x}}$$

Theorem 4. *The derivative of a product of two differentiable functions is equal to the product of the derivative of the first function by the second function plus the product of the first function by the derivative of the second function; that is,*

$$\textit{if } y = uv, \textit{ then } y' = u'v + uv'. \qquad \text{(VII)}$$

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* The expression $y = u(x) - v(x)$ is equivalent to $y = u(x) + (-1)v(x)$ and $y' = [u(x) + (-1)v(x)]' = u'(x) + [-v(x)]' = u'(x) - v'(x)$.

Proof. Reasoning as in the proof of the preceding theorem, we get

$$y = uv$$

$$y + \Delta y = (u + \Delta u)(v + \Delta v)$$

$$\Delta y = (u + \Delta u)(v + \Delta v) - uv = \Delta uv + u\Delta v + \Delta u\, \Delta v$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} v + u \frac{\Delta v}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}$$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} v + \lim_{\Delta x \to 0} u \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \frac{\Delta v}{\Delta x}$$

$$= \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right) v + u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + \lim_{\Delta x \to 0} \Delta u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

(since u and v are independent of Δx).

Let us consider the last term on the right-hand side:

$$\lim_{\Delta x \to 0} \Delta u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

Since $u(x)$ is a differentiable function, it is continuous. Consequently, $\lim\limits_{\Delta x \to 0} \Delta u = 0$. Also,

$$\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v' \neq \infty$$

Thus, the term under consideration is zero and we finally get

$$y' = u'v + uv'$$

The theorem just proved readily gives us the rule for differentiating the product of any number of functions.

Thus, if we have a product of three functions

$$y = uvw$$

then, by representing the right-hand side as the product of u and (vw), we get $y' = u'(vw) + u(vw)' = u'vw + u(v'w + vw') = u'vw + uv'w + uvw'$.

In this way we can obtain a similar formula for the derivative of the product of any (finite) number of functions. Namely, if $y = u_1 u_2 \ldots u_n$, then

$$y' = u_1' u_2 \ldots u_{n-1} u_n + u_1 u_2' \ldots u_{n-1} u_n + \ldots + u_1 u_2 \ldots u_{n-1} u_n'$$

Example 3. If $y = x^2 \sin x$, then

$$y' = (x^2)' \sin x + x^2 (\sin x)' = 2x \sin x + x^2 \cos x$$

Example 4. If $y = \sqrt{x} \sin x \cos x$, then

$$y' = (\sqrt{x})' \sin x \cos x + \sqrt{x} (\sin x)' \cos x + \sqrt{x} \sin x (\cos x)'$$

$$= \frac{1}{2\sqrt{x}} \sin x \cos x + \sqrt{x} \cos x \cos x + \sqrt{x} \sin x (-\sin x)$$

$$= \frac{1}{2\sqrt{x}} \sin x \cos x + \sqrt{x} (\cos^2 x - \sin^2 x) = \frac{\sin 2x}{4\sqrt{x}} + \sqrt{x} \cos 2x$$

Theorem 5. *The derivative of a fraction (that is, the quotient obtained by the division of two functions) is equal to a fraction whose denominator is the square of the denominator of the given fraction, and the numerator is the difference between the product of the denominator by the derivative of the numerator, and the product of the numerator by the derivative of the denominator; i.e.,*

$$\text{if } y=\frac{u}{v}, \text{ then } y'=\frac{u'v-uv'}{v^2}. \tag{VIII}$$

Proof. If Δy, Δu, and Δv are increments of the functions y, u, and v, corresponding to the increment Δx of the argument x, then

$$y+\Delta y=\frac{u+\Delta u}{v+\Delta v}$$

$$\Delta y=\frac{u+\Delta u}{v+\Delta v}-\frac{u}{v}=\frac{v\,\Delta u-u\,\Delta v}{v\,(v+\Delta v)}$$

$$\frac{\Delta y}{\Delta x}=\frac{\dfrac{v\,\Delta u-u\,\Delta v}{\Delta x}}{v\,(v+\Delta v)}=\frac{\dfrac{\Delta u}{\Delta x}v-u\dfrac{\Delta v}{\Delta x}}{v\,(v+\Delta v)}$$

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\dfrac{\Delta u}{\Delta x}v-u\dfrac{\Delta v}{\Delta x}}{v\,(v+\Delta v)}=\frac{v\lim\limits_{\Delta x\to 0}\dfrac{\Delta u}{\Delta x}-u\lim\limits_{\Delta x\to 0}\dfrac{\Delta v}{\Delta x}}{v\lim\limits_{\Delta x\to 0}(v+\Delta v)}$$

Whence, noting that $\Delta v\longrightarrow 0$ as $\Delta x\longrightarrow 0$,* we get

$$y'=\frac{u'v-uv'}{v^2}$$

Example 5. If $y=\dfrac{x^3}{\cos x}$, then

$$y'=\frac{(x^3)'\cos x-x^3(\cos x)'}{\cos^2 x}=\frac{3x^2\cos x+x^3\sin x}{\cos^2 x}$$

Note. If we have a function of the form

$$y=\frac{u(x)}{C}$$

where the denominator C is a constant, then in differentiating this function we do not need to use formula (VIII); it is better to make use of formula (V):

$$y'=\left(\frac{1}{C}u\right)'=\frac{1}{C}u'=\frac{u'}{C}$$

Of course, the same result is obtained if formula (VIII) is applied.

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*$\lim\limits_{\Delta x\to 0}\Delta v=0$ since $v(x)$ is a differentiable and, consequently, continuous function.

Example 6. If $y=\frac{\cos x}{7}$, then

$$y'=\frac{(\cos x)'}{7}=-\frac{\sin x}{7}$$

THE DERIVATIVE OF A LOGARITHMIC FUNCTION

Theorem. *The derivative of the function* $\log_a x$ *is* $\frac{1}{x}\log_a e$, *that is,*

$$\textit{if } y=\log_a x, \textit{ then } y'=\frac{1}{x}\log_a e \qquad \text{(IX)}$$

Proof. If Δy is an increment in the function $y=\log_a x$ that corresponds to the increment Δx in the argument x, then

$$y+\Delta y=\log_a(x+\Delta x)$$

$$\Delta y=\log_a(x+\Delta x)-\log_a x=\log_a\frac{x+\Delta x}{x}=\log_a\left(1+\frac{\Delta x}{x}\right)$$

$$\frac{\Delta y}{\Delta x}=\frac{1}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)$$

Multiply and divide by x the expression on the right-hand side of the latter equation:

$$\frac{\Delta y}{\Delta x}=\frac{1}{x}\frac{x}{\Delta x}\log_a\left(1+\frac{\Delta x}{x}\right)=\frac{1}{x}\log_a\left(1+\frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}$$

We denote the quantity $\frac{\Delta x}{x}$ by α. Obviously, $\alpha\rightarrow 0$ for the given x, and as $\Delta x\rightarrow 0$. Consequently,

$$\frac{\Delta y}{\Delta x}=\frac{1}{x}\log_a(1+\alpha)^{\frac{1}{\alpha}}$$

But, as we know from Sec. 2.7,

$$\lim_{\alpha\rightarrow\infty}(1+\alpha)^{\frac{1}{\alpha}}=e$$

But if the expression under the sign of the logarithm approaches the number e, then the logarithm of this expression approaches $\log_a e$ (in virtue of the continuity of the logarithmic function). We therefore finally get

$$y'=\lim_{\Delta x\rightarrow 0}\frac{\Delta y}{\Delta x}=\lim_{\alpha\rightarrow 0}\frac{1}{x}\log_a(1+\alpha)^{\frac{1}{\alpha}}=\frac{1}{x}\log_a e$$

Noting that $\log_a e=\frac{1}{\ln a}$, we can rewrite the formula as follows:

$$y'=\frac{1}{x}\frac{1}{\ln a}$$

The following is an important particular case of this formula: if $a=e$, then $\ln a=\ln e=1$; that is,

$$\textit{if } y=\ln x, \textit{ then } y'=\frac{1}{x}. \tag{X}$$

THE DERIVATIVE OF A COMPOSITE FUNCTION

Given a composite function $y=f(x)$, that is, such that it may be represented in the following form:

$$y=F(u), \quad u=\varphi(x)$$

or $y=F[\varphi(x)]$ (see Sec. 1.8). In the expression $y=F(u)$, u is called the *intermediate argument*.

Let us establish a rule for differentiating composite functions.

Theorem. *If a function $u=\varphi(x)$ has, at some point x, a derivative $u'_x=\varphi'(x)$, and the function $y=F(u)$ has, at the corresponding value of u, the derivative $y'_u=F'(u)$, then the composite function $y=F[\varphi(x)]$ at the given point x also has a derivative, which is equal to*

$$y'_x=F'_u(u)\,\varphi'(x)$$

where for u we must substitute the expression $u=\varphi(x)$. Briefly,

$$y'_x=y'_u u'_x$$

In other words, the derivative of a composite function is equal to the product of the derivative of the given function with respect to the intermediate argument u by the derivative of the intermediate argument with respect to x.

Proof. For a definite value of x we will have

$$u=\varphi(x), \quad y=F(u)$$

For the increased value of the argument $x+\Delta x$,

$$u+\Delta u=\varphi(x+\Delta x), \quad y+\Delta y=F(u+\Delta u)$$

Thus, to the increment Δx there corresponds an increment Δu, to which corresponds an increment Δy, whereby $\Delta u \to 0$ and $\Delta y \to 0$ as $\Delta x \to 0$. It is given that

$$\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}=y'_u$$

From this relation (taking advantage of the definition of a limit) we get (for $\Delta u \neq 0$)

$$\frac{\Delta y}{\Delta u}=y'_u+\alpha \tag{1}$$

where $\alpha \longrightarrow 0$ as $\Delta u \longrightarrow 0$. We rewrite (1) as

$$\Delta y = y'_u \Delta u + \alpha \Delta u \tag{2}$$

Equation (2) also holds true when $\Delta u = 0$ for an arbitrary α, since it turns into an identity, $0 = 0$. For $\Delta u = 0$ we shall assume $\alpha = 0$. Divide all terms of (2) by Δx:

$$\frac{\Delta y}{\Delta x} = y'_u \frac{\Delta u}{\Delta x} + \alpha \frac{\Delta u}{\Delta x} \tag{3}$$

It is given that

$$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u'_x, \quad \lim_{\Delta x \to 0} \alpha = 0$$

Passing to the limit as $\Delta x \longrightarrow 0$ in (3), we get

$$y'_x = y'_u u'_x \tag{4}$$

Example 1. Given a function $y = \sin(x^2)$. Find y'_x. Represent the given function as a function of a function as follows:

$$y = \sin u, \quad u = x^2$$

We find

$$y'_u = \cos u, \quad u'_x = 2x$$

Hence, by formula (4),

$$y'_x = y'_u u'_x = \cos u \cdot 2x$$

Replacing u by its expression, we finally get

$$y'_x = 2x \cos(x^2)$$

Example 2. Given the function $y = (\ln x)^3$. Find y'_x.
Represent this function as follows:

$$y = u^3, \quad u = \ln x$$

We find

$$y'_u = 3u^2; \quad u'_x = \frac{1}{x}$$

Hence,

$$y'_x = 3u^2 \frac{1}{x} = 3(\ln x)^2 \frac{1}{x}$$

If a function $y = f(x)$ is such that it may be represented in the form

$$y = F(u), \quad u = \varphi(v), \quad v = \psi(x)$$

the derivative y'_x, is found by a successive application of the foregoing theorem.

Applying the proved rule, we have

$$y'_x = y'_u u'_x$$

Applying the same theorem to find u'_x, we have

$$u'_x = u'_v v'_x$$

Substituting the expression of u'_x into the preceding equation, we get

$$y'_x = y'_u u'_v v'_x \tag{5}$$

or

$$y'_x = F'_u(u)\,\varphi'_v(v)\,\psi'_x(x)$$

Example 3. Given the function $y = \sin[(\ln x)^3]$. Find y'_x. Represent the function as follows:

$$y = \sin u, \quad u = v^3, \quad v = \ln x$$

We then find

$$y'_u = \cos u, \quad u'_v = 3v^2, \quad v'_x = \frac{1}{x}$$

In this way, by formula (5), we get

$$y'_x = y'_u u'_v v'_x = 3(\cos u)\, v^2 \frac{1}{x}$$

or, finally,

$$y'_x = \cos[(\ln x)^3] \cdot 3(\ln x)^2 \frac{1}{x}$$

It is to be noted that the function considered is defined only for $x > 0$.

DERIVATIVES OF THE FUNCTIONS $y = \tan x$, $y = \cot x$, $y = \ln|x|$

Theorem 1. *The derivative of the function* $\tan x$ *is* $\frac{1}{\cos^2 x}$, *i. e.*,

$$\text{if } y = \tan x, \quad \text{then } y' = \frac{1}{\cos^2 x}. \tag{XI}$$

Proof. Since

$$y = \frac{\sin x}{\cos x}$$

by the rule for differentiating a fraction [see formula (VIII), Sec. 3.7] we get

$$y' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Theorem 2. *The derivative of the function* $\cot x$ *is* $-\frac{1}{\sin^2 x}$, *i. e.*,

$$\text{if } y = \cot x, \text{ then } y' = -\frac{1}{\sin^2 x}. \tag{XII}$$

Proof. Since $y=\frac{\cos x}{\sin x}$, we have

$$y'=\frac{(\cos x)'\sin x-\cos x(\sin x)'}{\sin^2 x}=\frac{-\sin x\sin x-\cos x\cos x}{\sin^2 x}$$
$$=-\frac{\sin^2 x+\cos^2 x}{\sin^2 x}=-\frac{1}{\sin^2 x}$$

Example 1. If $y=\tan\sqrt{x}$, then

$$y'=\frac{1}{\cos^2\sqrt{x}}(\sqrt{x})'=\frac{1}{2\sqrt{x}}\,\frac{1}{\cos^2\sqrt{x}}$$

Example 2. If $y=\ln\cot x$, then

$$y'=\frac{1}{\cot x}(\cot x)'=\frac{1}{\cot x}\left(-\frac{1}{\sin^2 x}\right)=-\frac{1}{\cos x\sin x}=-\frac{2}{\sin 2x}$$

Theorem 3. *The derivative of the function* $\ln|x|$ (Fig. 63) *is* $\frac{1}{x}$, *i.e.*,

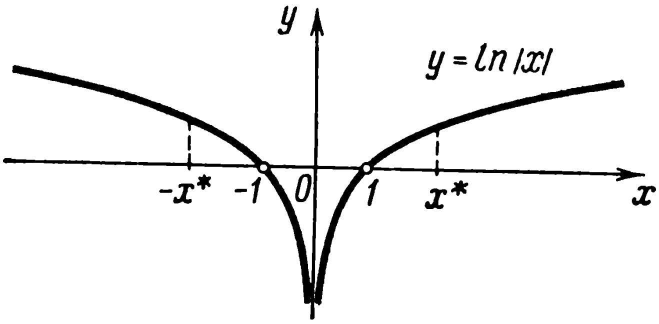

Fig. 63

if $y=\ln|x|$,

then $y'=\frac{1}{x}$. (XIII)

Proof. (a) If $x>0$, then $|x|=x$, $\ln|x|=\ln x$, and therefore

$$y'=\frac{1}{x}$$

(b) Let $x<0$. Then $|x|=-x$. But

$$\ln|x|=\ln(-x)$$

(It will be noted that if $x<0$, then $-x>0$.) Let us represent the function $y=\ln(-x)$ as a composite function by putting

$$y=\ln u,\quad u=-x$$

Then

$$y'_x=y'_u u'_x=\frac{1}{u}(-1)=\frac{1}{-x}(-1)=\frac{1}{x}$$

And so for negative values of x we also have the equation

$$y'_x=\frac{1}{x}$$

Hence, formula (XIII) has been proved for any value of $x\neq 0$. (For $x=0$ the function $\ln|x|$ is not defined.)