Lecture 6

THE NUMBER *e*

Let us consider the variable

$$\left(1+\frac{1}{n}\right)^n$$

where n is an increasing variable that takes on the values 1, 2, 3, ...

Theorem 1. *The variable* $\left(1+\frac{1}{n}\right)^n$, *as* $n \longrightarrow \infty$, *has a limit between the numbers* 2 *and* 3.

Proof. By Newton's binomial formula we have

$$\left(1+\frac{1}{n}\right)^n = 1+\frac{n}{1}\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\cdot\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\left(\frac{1}{n}\right)^3$$
$$\ldots+\frac{n(n-1)(n-2)\ldots[n-(n-1)]}{1\cdot 2\cdot\ldots\cdot n}\left(\frac{1}{n}\right)^n \qquad (1)$$

Carrying out the obvious algebraic manipulations in (1), we get

$$\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{1\cdot 2}\left(1-\frac{1}{n}\right)+\frac{1}{1\cdot 2\cdot 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$
$$\ldots+\frac{1}{1\cdot 2\cdot\ldots\cdot n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right) \qquad (2)$$

From the latter equality it follows that the variable $\left(1+\frac{1}{n}\right)^n$ is an increasing variable as n increases.

Indeed, when passing from the value n to the value $n+1$, each term in the latter sum increases,

$$\frac{1}{1\cdot 2}\left(1-\frac{1}{n}\right)<\frac{1}{1\cdot 2}\left(1-\frac{1}{n+1}\right) \text{ and so forth,}$$

and another term is added. (All terms of the expansion are positive.)

We shall show that the variable $\left(1+\frac{1}{n}\right)^n$ is bounded. Noting that $\left(1-\frac{1}{n}\right)<1$, $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)<1$, etc., we obtain from expression (2) the inequality

$$\left(1+\frac{1}{n}\right)^n<1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\ldots+\frac{1}{1\cdot 2\cdot 3\cdot\ldots\cdot n}$$

Further noting that

$$\frac{1}{1\cdot 2\cdot 3}<\frac{1}{2^2},\ \frac{1}{1\cdot 2\cdot 3\cdot 4}<\frac{1}{2^3},\ \ldots,\ \frac{1}{1\cdot 2\cdot\ldots\cdot n}<\frac{1}{2^{n-1}}$$

we can write the inequality

$$\left(1+\frac{1}{n}\right)^n<1+\underbrace{1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}}$$

The grouped terms on the right-hand side of this inequality form a geometric progression with common ratio $q=\frac{1}{2}$ and the first term $a=1$, and so

$$\left(1+\frac{1}{n}\right)^n<1+\left[1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}\right]$$
$$=1+\frac{a-aq^n}{1-q}=1+\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}=1+\left[2-\left(\frac{1}{2}\right)^{n-1}\right]<3$$

Consequently, for all n we get

$$\left(1+\frac{1}{n}\right)^n<3 \quad .$$

From (2) it follows that

$$\left(1+\frac{1}{n}\right)^n\geqslant 2$$

Thus, we get the inequality

$$2\leqslant\left(1+\frac{1}{n}\right)^n<3 \tag{3}$$

This proves that the variable $\left(1+\frac{1}{n}\right)^n$ is bounded.

Thus, the variable $\left(1+\frac{1}{n}\right)^n$ is an increasing and bounded variable; therefore, by Theorem 7, Sec. 2.5, it has a limit. This limit is denoted by the letter e.

Definition. The limit of the variable $\left(1+\frac{1}{n}\right)^n$ as $n\to\infty$ is the *number e*:

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n*$$

By Theorem 6, Sec. 2.5, it follows from inequality (3) that the number e satisfies the inequality $2\leqslant e\leqslant 3$. The theorem is thus proved.

The number e is an irrational number. Later on, a method will be shown that permits calculating e to any degree of accuracy. Its value to ten decimal places is

$$e=2.7182818284...$$

Theorem 2. *The function* $\left(1+\frac{1}{x}\right)^x$ *approaches the limit e as x approaches infinity,* $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$.

Proof. It has been shown that $\left(1+\frac{1}{n}\right)^n\to e$ as $n\to\infty$, if n takes on positive integral values. Now let x approach infinity while taking on both fractional and negative values.

(1) Let $x\to+\infty$. Each of its values lies between two positive integers,

$$n\leqslant x<n+1$$

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* It may be shown that $\left(1+\frac{1}{n}\right)^n\to e$ as $n\to+\infty$ even if n is not an increasing variable quantity.

The following inequalities will be fulfilled:

$$\frac{1}{n} \geqslant \frac{1}{x} > \frac{1}{n+1}$$

$$1+\frac{1}{n} \geqslant 1+\frac{1}{x} > 1+\frac{1}{n+1}$$

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{x}\right)^{x} > \left(1+\frac{1}{n+1}\right)^{n}$$

If $x \to \infty$, it is obvious that $n \to \infty$. Let us find the limits of the variables between which the variable $\left(1+\frac{1}{x}\right)^{x}$ lies:

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)$$

$$= \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n} \cdot \lim_{n\to+\infty}\left(1+\frac{1}{n}\right) = e \cdot 1 = e$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n} = \lim_{n\to+\infty}\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}}$$

$$= \frac{\lim\limits_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim\limits_{n\to+\infty}\left(1+\frac{1}{n+1}\right)} = \frac{e}{1} = e$$

Hence, by Theorem 4, Sec. 2.5,

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = e \tag{4}$$

(2) Let $x \to -\infty$. We introduce a new variable $t = -(x+1)$ or $x = -(t+1)$. When $t \to +\infty$, then $x \to -\infty$. We can write

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{t\to+\infty}\left(1-\frac{1}{t+1}\right)^{-t-1} = \lim_{t\to+\infty}\left(\frac{t}{t+1}\right)^{-t-1}$$

$$= \lim_{t\to+\infty}\left(\frac{t+1}{t}\right)^{t+1} = \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{t+1}$$

$$= \lim_{t\to+\infty}\left(1+\frac{1}{t}\right)^{t}\left(1+\frac{1}{t}\right) = e \cdot 1 = e$$

The theorem is proved. The graph of the function $y = \left(1+\frac{1}{x}\right)^{x}$ is shown in Fig. 45.

If in (4) we put $\frac{1}{x} = \alpha$, then as $x \to \infty$ we have $\alpha \to 0$ (but $\alpha \neq 0$) and we get

$$\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}} = e$$

Examples:

$$(1)\ \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+5}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{5}$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{5}=e\cdot 1=e.$$

$$(2)\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{x}\left(1+\frac{1}{x}\right)^{x}$$

$$=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\cdot\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\cdot\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e\cdot e\cdot e=e^{3}.$$

$$(3)\ \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{2y}=e^{2}.$$

$$(4)\ \lim_{x\to\infty}\left(\frac{x+3}{x-1}\right)^{x+3}=\lim_{x\to\infty}\left(\frac{x-1+4}{x-1}\right)^{x+3}=\lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{x+3}$$

$$=\lim_{x\to\infty}\left(1+\frac{4}{x-1}\right)^{(x-1)+4}=\lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y+4}$$

$$=\lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y}\cdot\lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{4}=e^{4}\cdot 1=e^{4}.$$

Note. The exponential function e^x plays a very important role in mathematics, mechanics (oscillation theory), electrical and radio

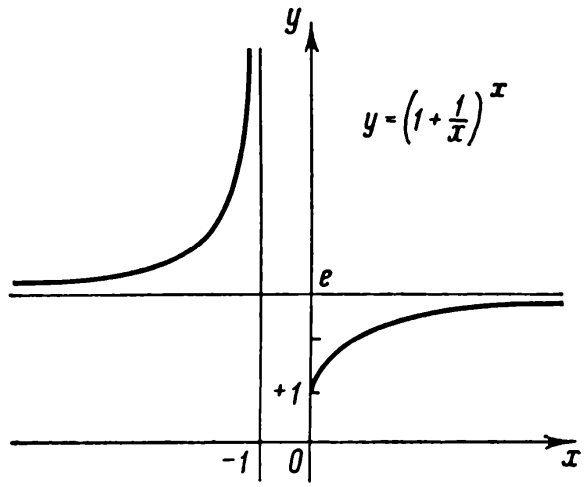

Fig. 45

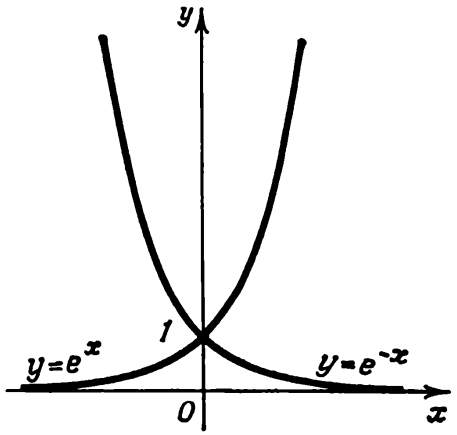

Fig. 46

engineering, radiochemistry, etc. The graphs of the functions $y=e^x$ and $y=e^{-x}$ are shown in Fig. 46.

NATURAL LOGARITHMS

Earlier we defined the logarithmic function $y=\log_a x$. The number a is called the base of the logarithms. If $a=10$, then y is the base-10 (common) logarithm of the number x and is denoted $y=\log x$. In school courses of mathematics we have

tables of common logarithms, which are called Briggs' logarithms after the English mathematician Briggs (1561-1630).

Logarithms to the base $e=2.71828...$ are called *natural* or *Napierian logarithms* after one of the first inventors of logarithmic tables, the Scotch mathematician Napier (1550-1617).* Therefore, if $e^y=x$, then y is called the natural logarithm of the number x. In writing we have $y=\ln x$ (after the initial letters of *logarithmus naturalis*) in place of $y=\log_e x$. Graphs of the function $y=\ln x$ and $y=\log x$ are plotted in Fig. 47.

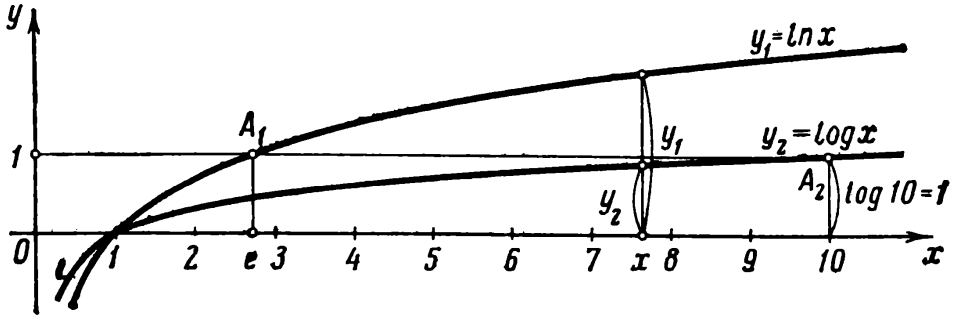

Fig. 47

Let us now establish a relationship between common and natural logarithms of one and the same number x. Let $y=\log x$ or $x=10^y$. We take logarithms of the left and right sides of the latter equality to the base e and get $\ln x=y\ln 10$. We find $y=\frac{1}{\ln 10}\ln x$, or substituting the value of y, we have $\log x=\frac{1}{\ln 10}\ln x$.

Thus, if we know the natural logarithm of a number x, the common logarithm of this number is found by multiplying by the factor $M=\frac{1}{\ln 10}\approx 0.434294$, which factor is independent of x. The number M is the *modulus* of common logarithms with respect to natural logarithms:

$$\log x=M\ln x$$

If in this identity we put $x=e$, we obtain an expression of the number M in terms of common logarithms:

$$\log e=M\ (\ln e=1)$$

Natural logarithms are expressed in terms of common logarithms as follows:

$$\ln x=\frac{1}{M}\log x$$

where

$$\frac{1}{M}\approx 2.302585$$

* The first logarithmic tables were constructed by the Swiss mathematician Bürgi (1552-1632) to a base close to the number e.

CONTINUITY OF FUNCTIONS

Let a function $y=f(x)$ be defined for some value x_0 and in some neighbourhood with centre at x_0. Let $y_0=f(x_0)$.

If x receives some positive or negative (it is immaterial which) increment Δx and assumes the value $x=x_0+\Delta x$, then the function y too will receive an increment Δy. The new increased value of the function will be $y_0+\Delta y=f(x_0+\Delta x)$ (Fig. 48). The increment of the function Δy will be expressed by the formula

$$\Delta y=f(x_0+\Delta x)-f(x_0)$$

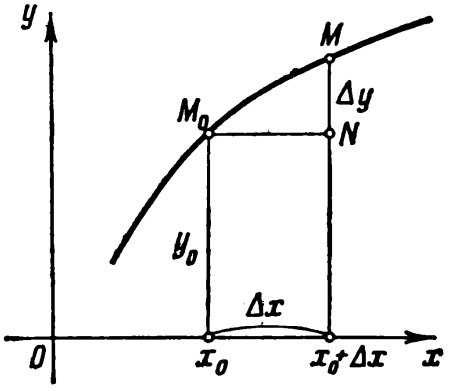

Fig. 48

Definition 1. A function $y=f(x)$ is called *continuous for the value* $x=x_0$ (or *at the point* x_0) if it is defined in some neighbourhood of the point x_0 (obviously, at the point x_0 as well) and if

$$\lim_{\Delta x\to 0}\Delta y=0 \tag{1}$$

or, which is the same thing,

$$\lim_{\Delta x\to 0}[f(x_0+\Delta x)-f(x_0)]=0 \tag{2}$$

The continuity condition (2) may also be written as follows:

$$\lim_{\Delta x\to 0}f(x_0+\Delta x)=f(x_0)$$

or

$$\lim_{x\to x_0}f(x)=f(x_0) \tag{3}$$

but

$$x_0=\lim_{x\to x_0}x$$

Hence, (3) may be written thus:

$$\lim_{x\to x_0}f(x)=f\left(\lim_{x\to x_0}x\right) \tag{4}$$

In other words, in order to find the limit of a continuous function as $x\to x_0$, it is sufficient, in the expression of the function, to put the value x_0 in place of the argument x.

In descriptive geometrical terms, the continuity of a function at a given point signifies that the difference of the ordinates on the graph of the function $y=f(x)$ at the points $x_0+\Delta x$ and x_0 will, in absolute value, be arbitrarily small, provided $|\Delta x|$ is sufficiently small.

Example 1. We shall prove that the function $y=x^2$ is continuous at an arbitrary point x_0. Indeed,

$$y_0=x_0^2,\ y_0+\Delta y=(x_0+\Delta x)^2,\ \Delta y=(x_0+\Delta x)^2-x_0^2=2x_0\Delta x+\Delta x^2,$$
$$\lim_{\Delta x\to 0}\Delta y=\lim_{\Delta x\to 0}(2x_0\Delta x+\Delta x^2)=2x_0\lim_{\Delta x\to 0}\Delta x+\lim_{\Delta x\to 0}\Delta x\cdot\lim_{\Delta x\to 0}\Delta x=0$$

for any way that Δx may approach zero (Figs. 49*a* and 49*b*).

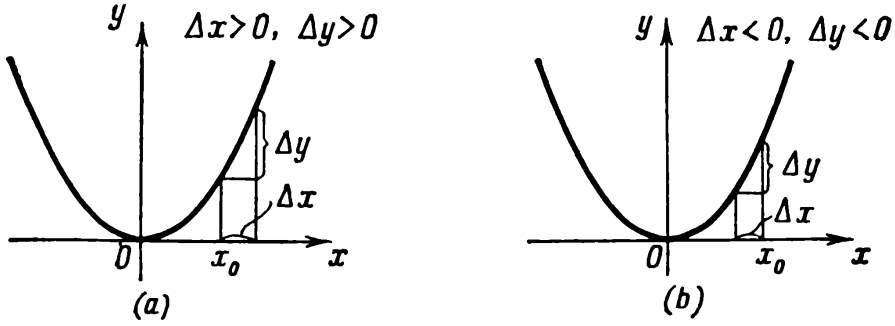

Fig. 49

Example 2. We shall prove that the function $y=\sin x$ is continuous at an arbitrary point x_0. Indeed,

$$y_0=\sin x_0,\quad y_0+\Delta y=\sin(x_0+\Delta x),$$
$$\Delta y=\sin(x_0+\Delta x)-\sin x_0=2\sin\frac{\Delta x}{2}\cdot\cos\left(x_0+\frac{\Delta x}{2}\right)$$

It has been shown that $\lim\limits_{\Delta x\to 0}\sin\frac{\Delta x}{2}=0$ (Example 7, Sec. 2.5). The function $\cos\left(x+\frac{\Delta x}{2}\right)$ is bounded. Therefore, $\lim\limits_{\Delta x\to 0}\Delta y=0$.

In similar fashion, by considering each basic elementary function, it is possible to prove that each basic elementary function is continuous at every point at which it is defined.

We will now prove the following theorem.

Theorem 1. *If the functions $f_1(x)$ and $f_2(x)$ are continuous at a point x_0, then the sum $\psi(x)=f_1(x)+f_2(x)$ is also a function continuous at the point x_0.*

Proof. Since $f_1(x)$ and $f_2(x)$ are continuous, on the basis of (3) we can write

$$\lim_{x\to x_0}f_1(x)=f_1(x_0)\quad\text{and}\quad\lim_{x\to x_0}f_2(x)=f_2(x_0)$$

By Theorem 1 on limits, we can write

$$\lim_{x\to x_0}\psi(x)=\lim_{x\to x_0}[f_1(x)+f_2(x)]$$
$$=\lim_{x\to x_0}f_1(x)+\lim_{x\to x_0}f_2(x)=f_1(x_0)+f_2(x_0)=\psi(x_0)$$

Thus, the sum $\psi(x)=f_1(x)+f_2(x)$ is a continuous function. The proof is complete.

Note, as a corollary, that the theorem holds true for any finite number of terms.

Using the properties of limits, we can also prove the following theorems:

(a) *The product of two continuous functions is a continuous function.*

(b) *The quotient of two continuous functions is a continuous function if the denominator does not vanish at the point under consideration.*

(c) *If* $u=\varphi(x)$ *is continuous at* $x=x_0$ *and* $f(u)$ *is continuous at the point* $u_0=\varphi(x_0)$, *then the composite function* $f[\varphi(x)]$ *is continuous at the point* x_0.

Using these theorems, we can prove the following theorem.

Theorem 2. *Every elementary function is continuous at every point at which it is defined.**

Example 3. The function $y=x^2$ is continuous at every point x_0 and therefore

$$\lim_{x\to x_0} x^2=x_0^2$$

$$\lim_{x\to 3} x^2=3^2=9$$

Example 4. The function $y=\sin x$ is continuous at every point and therefore

$$\lim_{x\to\frac{\pi}{4}} \sin x=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

Example 5. The function $y=e^x$ is continuous at every point and therefore $\lim\limits_{x\to a} e^x=e^a$.

Example 6. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=\lim\limits_{x\to 0}\dfrac{1}{x}\ln(1+x)=\lim\limits_{x\to 0}\ln\left[(1+x)^{\frac{1}{x}}\right]$. Since $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$ and the function $\ln z$ is continuous for $z>0$ and, consequently, for $z=e$,

$$\lim_{x\to 0}\ln\left[(1+x)^{\frac{1}{x}}\right]=\ln\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]=\ln e=1$$

Definition 2. If a function $y=f(x)$ is continuous at each point of a certain interval (a, b), where $a<b$, then it is said that the function *is continuous in this interval.*

If the function is also defined for $x=a$ and $\lim\limits_{x\to a+0} f(x)=f(a)$, it is said that $f(x)$ at the point $x=a$ *is continuous on the right.*

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* This problem is discussed in detail in G. M. Fikhtengolts' *Fundamentals of Mathematical Analysis*, Vol. 1, Fizmatgiz, Moscow, 1968 (in Russian).

If $\lim\limits_{x \to b-0} f(x) = f(b)$, it is said that the function $f(x)$ at the point $x = b$ *is continuous on the left.*

If the function $f(x)$ is continuous at each point of the interval (a, b) and is continuous at the end points of the interval, on the right and left, respectively, then we say that the function $f(x)$ *is continuous over the closed interval* $[a, b]$.

Example 7. The function $y = x^3$ is continuous in any closed interval $[a, b]$. This follows from Example 1.

If at some point $x = x_0$ at least one of the conditions of continuity is not fulfilled for the function $y = f(x)$, that is, if for $x = x_0$ the function is not defined or there does not exist a limit $\lim\limits_{x \to x_0} f(x)$ or $\lim\limits_{x \to x_0} f(x) \neq f(x_0)$ in the arbitrary approach of $x \to x_0$, although the expressions on the right and left exist, then at $x = x_0$ the function $y = f(x)$ is *discontinuous*. In this case, the point $x = x_0$ is called the *point of discontinuity* of the function.

Example 8. The function $y = \frac{1}{x}$ is discontinuous at $x = 0$. Indeed, the function is not defined at $x = 0$.

$$\lim_{x \to 0+0} \frac{1}{x} = +\infty, \quad \lim_{x \to 0-0} \frac{1}{x} = -\infty$$

It is easy to show that this function is continuous for any value $x \neq 0$.

Example 9. The function $y = 2^{\frac{1}{x}}$ is discontinuous at $x = 0$. Indeed, $\lim\limits_{x \to 0+0} 2^{\frac{1}{x}} = \infty$, $\lim\limits_{x \to 0-0} 2^{\frac{1}{x}} = 0$. The function is not defined at $x = 0$ (Fig. 50).

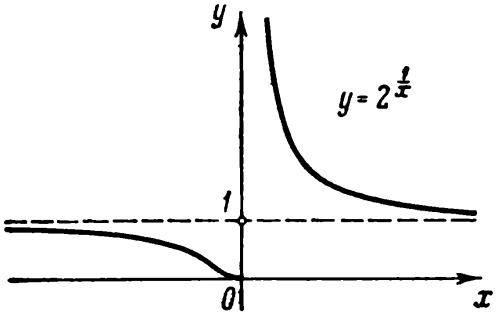

Fig. 50

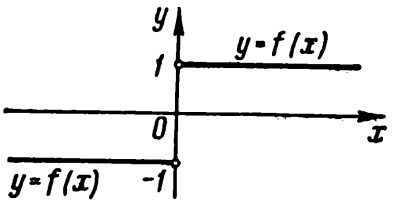

Fig. 51

Example 10. Consider the function $f(x) = \frac{x}{|x|}$. For $x < 0$, $\frac{x}{|x|} = -1$, for $x > 0$, $\frac{x}{|x|} = 1$. Hence,

$$\lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} \frac{x}{|x|} = -1,$$

$$\lim_{x \to 0+0} f(x) = \lim_{x \to 0+0} \frac{x}{|x|} = 1$$

the function is not defined at $x=0$. We have thus established the fact that the function $f(x)=\frac{x}{|x|}$ is discontinuous at $x=0$ (Fig. 51).

Example 11. The earlier examined function (Example 4, Sec. 2.3) $y=\sin(1/x)$ is discontinuous at $x=0$.

Definition 3. If the function $f(x)$ is such that there exist finite limits $\lim\limits_{x\to x_0+0} f(x)=f(x_0+0)$ and $\lim\limits_{x\to x_0-0} f(x)=f(x_0-0)$, but either $\lim\limits_{x\to x_0+0} f(x)\neq\lim\limits_{x\to x_0-0} f(x)$ or the value of the function $f(x)$ at $x=x_0$ is not defined, then $x=x_0$ is called a *point of discontinuity of the first kind.* (For example, for the function considered in Example 10, the point $x=0$ is a point of discontinuity of the first kind.)

CERTAIN PROPERTIES OF CONTINUOUS FUNCTIONS

In this section we shall consider a number of properties of functions that are continuous on an interval. These properties will be stated in the form of theorems given without proof.*

Theorem 1. *If a function $y=f(x)$ is continuous on some interval $[a, b]$ $(a\leqslant x\leqslant b)$, there will be, on this interval, at least one point $x=x_1$ such that the value of the function at that point will satisfy the relation*

$$f(x_1)\geqslant f(x)$$

where x is any other point of the interval, and there will be at least one point x_2 such that the value of the function at that point will satisfy the relation

$$f(x_2)\leqslant f(x)$$

We shall call the value of the function $f(x_1)$ the *greatest value* of the function $y=f(x)$ on the interval $[a, b]$, and the value of the function $f(x_2)$ the *smallest (least) value* of the function on the interval $[a, b]$.

This theorem is briefly stated as follows:

A function continuous on the interval $a\leqslant x\leqslant b$ attains on this interval (at least once) a greatest value M and a smallest value m.

The meaning of this theorem is clearly illustrated in Fig. 52.

Note. The assertion that there exists a greatest value of the function may prove incorrect if one considers the values of the function in the interval $a<x<b$. For instance, if we consider the function $y=x$ in the interval $0<x<1$, there will be no greatest and no least values among them. Indeed, there is no least

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* These theorems are proved in G. M. Fikhtengolts' *Principles of Mathematical Analysis*, Vol. 1, Fizmatgiz, 1968 (in Russian).

value or greatest value of x in the interval. (There is no extreme left point, since no matter what point x^* we take there will be a point to the left of it, for instance, the point $\frac{x^*}{2}$; likewise, there is no extreme right point; consequently, there is no least and no greatest value of the function $y=x$.)

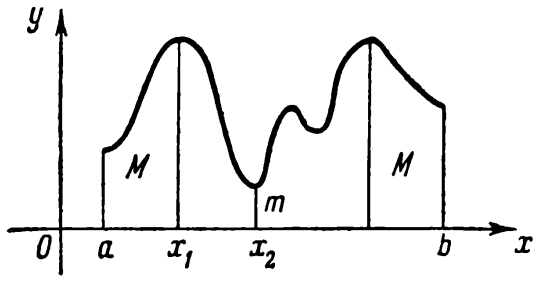

Fig. 52

Theorem 2. *Let the function $y=f(x)$ be continuous on the interval $[a, b]$ and at the end points of this interval let it take on values of different signs; then between the points a and b there will be at least one point $x=c$, at which the function becomes zero:*

$$f(c)=0,\ a<c<b$$

This theorem has a simple geometrical meaning. The graph of a continuous function $y=f(x)$ joining the points $M_1[a, f(a)]$ and $M_2[b, f(b)]$, where $f(a)<0$ and $f(b)>0$ or $f(a)>0$ and $f(b)<0$, cuts the x-axis in at least one point (Fig. 53).

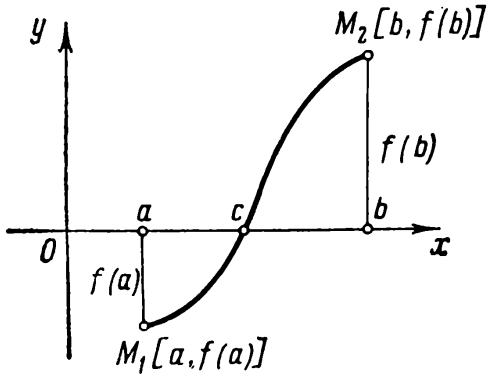

Fig. 53

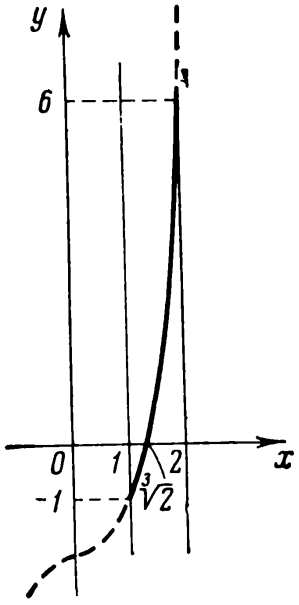

Fig. 54

Example. Given the function $y=x^3-2$; $y_{x=1}=-1$, $y_{x=2}=6$. It is continuous in the interval [1, 2]. Hence, in this interval there is a point where $y=x^3-2$ becomes zero. Indeed, $y=0$ when $x=\sqrt[3]{2}$ (Fig. 54).

Theorem 3. *Let a function $y=f(x)$ be defined and continuous in the interval $[a, b]$. If at the end points of this interval the function takes on unequal values $f(a)=A$, $f(b)=B$, then no matter what the number μ between numbers A and B, there will be a point $x=c$ between a and b such that $f(c)=\mu$.*

The meaning of this theorem is clearly illustrated in Fig. 55. In the given case, any straight line $y=\mu$ cuts the graph of the function $y=f(x)$.

Note. It will be noted that Theorem 2 is a particular case of this theorem, for if A and B have different signs, then for μ one can take 0, and then $\mu=0$ will lie between the numbers A and B.

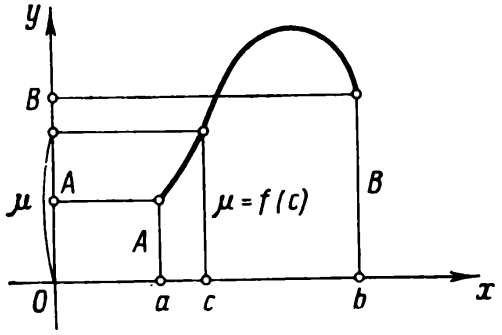

Fig. 55

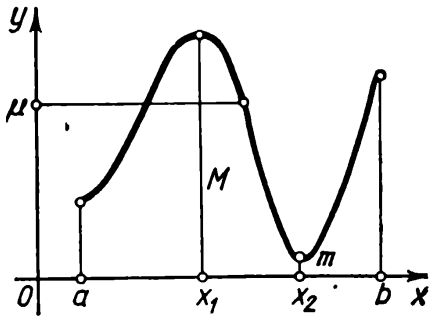

Fig. 56

Corollary of Theorem 3. *If a function $y=f(x)$ is continuous in some interval and takes on a greatest value and a least value, then in this interval it takes on, at least once, any value lying between the greatest and least values.*

Indeed, let $f(x_1)=M$, $f(x_2)=m$. Consider the interval $[x_1, x_2]$. By Theorem 3, in this interval the function $y=f(x)$ takes on any value μ lying between M and m. But the interval $[x_1, x_2]$ lies inside the interval under consideration in which the function $f(x)$ is defined (Fig. 56).

COMPARING INFINITESIMALS

Let several infinitesimal quantities

$$\alpha, \beta, \gamma, \ldots$$

be at the same time functions of one and the same argument x and let them approach zero as x approaches some limit a or infinity. We shall describe the approach of these variables to zero when we consider their ratios.*

We shall, in future, make use of the following definitions.

Definition 1. If the ratio $\frac{\beta}{\alpha}$ has a finite nonzero limit, that is, if $\lim\frac{\beta}{\alpha}=A\neq 0$, and therefore, $\lim\frac{\alpha}{\beta}=\frac{1}{A}\neq 0$, the infinitesimals β and α are called *infinitesimals of the same order*.

---

* We assume that the infinitesimal in the denominator does not vanish in some neighbourhood of the point a.

Example 1. Let $\alpha = x$, $\beta = \sin 2x$, where $x \to 0$. The infinitesimals α and β are of the same order because

$$\lim_{x \to 0} \frac{\beta}{\alpha} = \lim_{x \to 0} \frac{\sin 2x}{x} = 2$$

Example 2. When $x \to 0$, the infinitesimals x, $\sin 3x$, $\tan 2x$, $7 \ln (1+x)$ are infinitesimals of the same order. The proof is similar to that given in Example 1.

Definition 2. If the ratio of two infinitesimals $\frac{\beta}{\alpha}$ approaches zero, that is, if $\lim \frac{\beta}{\alpha} = 0$ $\left(\text{and } \lim \frac{\alpha}{\beta} = \infty\right)$, then the infinitesimal β is called *an infinitesimal of higher order than* α, and the infinitesimal α is called *an infinitesimal of lower order than* β.

Example 3. Let $\alpha = x$, $\beta = x^n$, $n > 1$, $x \to 0$. The infinitesimal β is an infinitesimal of higher order than the infinitesimal α since

$$\lim_{x \to 0} \frac{x^n}{x} = \lim_{x \to 0} x^{n-1} = 0$$

Here, the infinitesimal α is an infinitesimal of lower order than β.

Definition 3. An infinitesimal β is called *an infinitesimal of the kth order relative to an infinitesimal* α, if β and α^k are infinitesimals of the same order, that is, if $\lim \frac{\beta}{\alpha^k} = A \neq 0$.

Example 4. If $\alpha = x$, $\beta = x^3$, then as $x \to 0$ the infinitesimal β is an infinitesimal of the third order relative to the infinitesimal α, since

$$\lim_{x \to 0} \frac{\beta}{\alpha^3} = \lim_{x \to 0} \frac{x^3}{x^3} = 1$$

Definition 4. If the ratio of two infinitesimals $\frac{\beta}{\alpha}$ approaches unity, that is, if $\lim \frac{\beta}{\alpha} = 1$, the infinitesimals β and α are called *equivalent infinitesimals* and we write $\alpha \sim \beta$.

Example 5. Let $\alpha = x$ and $\beta = \sin x$, where $x \to 0$. The infinitesimals α and β are equivalent, since

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Example 6. Let $\alpha = x$, $\beta = \ln (1+x)$, where $x \to 0$. The infinitesimals α and β are equivalent, since

$$\lim_{x \to 0} \frac{\ln (1+x)}{x} = 1$$

(see Example 6, Sec. 2.9).

Theorem 1. *If* α *and* β *are equivalent infinitesimals, their difference* $\alpha - \beta$ *is an infinitesimal of higher order than* α *and than* β.

Proof. Indeed,

$$\lim \frac{\alpha-\beta}{\alpha}=\lim\left(1-\frac{\beta}{\alpha}\right)=1-\lim \frac{\beta}{\alpha}=1-1=0$$

Theorem 2. *If the difference of two infinitesimals $\alpha-\beta$ is an infinitesimal of higher order than α and than β, then α and β are equivalent infinitesimals.*

Proof. Let $\lim \frac{\alpha-\beta}{\alpha}=0$, then $\lim\left(1-\frac{\beta}{\alpha}\right)=0$, or $1-\lim \frac{\beta}{\alpha}=0$, or $1=\lim \frac{\beta}{\alpha}$, i.e., $\alpha \sim \beta$. If $\lim \frac{\alpha-\beta}{\beta}=0$, then $\lim\left(\frac{\alpha}{\beta}-1\right)=0$, $\lim \frac{\alpha}{\beta}=1$, that is, $\alpha \sim \beta$.

Example 7. Let $\alpha=x$, $\beta=x+x^3$, where $x \to 0$.

The infinitesimals α and β are equivalent, since their difference $\beta-\alpha=x^3$ is an infinitesimal of higher order than α and than β. Indeed,

$$\lim_{x\to 0} \frac{\beta-\alpha}{\alpha}=\lim_{x\to 0} \frac{x^3}{x}=\lim_{x\to 0} x^2=0$$

$$\lim_{x\to 0} \frac{\beta-\alpha}{\beta}=\lim_{x\to 0} \frac{x^3}{x+x^3}=\lim_{x\to 0} \frac{x^2}{1+x^2}=0$$

Example 8. For $x \to \infty$ the infinitesimals $\alpha=\frac{x+1}{x^2}$ and $\beta=\frac{1}{x}$ are equivalent infinitesimals, since their difference $\alpha-\beta=\frac{x+1}{x^2}-\frac{1}{x}=\frac{1}{x^2}$ is an infinitesimal of higher order than α and than β. The limit of the ratio of α and β is unity:

$$\lim_{x\to\infty} \frac{\alpha}{\beta}=\lim_{x\to\infty} \frac{\frac{x+1}{x^2}}{\frac{1}{x}}=\lim_{x\to\infty} \frac{x+1}{x}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)=1$$

Note. If the ratio of two infinitesimals $\frac{\beta}{\alpha}$ has no limit and does not approach infinity, then β and α are not comparable in the above sense.

Example 9. Let $\alpha=x$, $\beta=x\sin\frac{1}{x}$, where $x \to 0$. The infinitesimals α and β cannot be compared because their ratio $\frac{\beta}{\alpha}=\sin\frac{1}{x}$ as $x \to 0$ does not approach either a finite limit or infinity (see Example 4, Sec. 2.3).