Lecture 5

A FUNCTION THAT APPROACHES INFINITY. BOUNDED FUNCTIONS

We have considered cases when a function $f(x)$ approaches a certain limit b as $x \to a$ or as $x \to \infty$.

Let us now take the case where the function $y = f(x)$ approaches infinity when the argument varies in some way.

Definition 1. The function $f(x)$ approaches infinity as $x \to a$, i.e., it is an *infinitely large* quantity as $x \to a$ if for each positive number M, no matter how large, it is possible to find a $\delta > 0$ such that for all values of x different from a and satisfying the condition $|x-a| < \delta$, we have the inequality $|f(x)| > M$.

If $f(x)$ approaches infinity as $x \to a$, we write

$$\lim_{x \to a} f(x) = \infty$$

or $f(x) \to \infty$ as $x \to a$.

If $f(x)$ approaches infinity as $x \to a$ and, in the process, assumes only positive or only negative values, the appropriate notation is $\lim\limits_{x \to a} f(x) = +\infty$ or $\lim\limits_{x \to a} f(x) = -\infty$.

Example 1. We shall prove that $\lim\limits_{x \to 1} \frac{1}{(1-x)^2} = +\infty$. Indeed, for any $M > 0$ we have

$$\frac{1}{(1-x)^2} > M$$

provided

$$(1-x)^2 < \frac{1}{M}, \quad |1-x| < \frac{1}{\sqrt{M}} = \delta$$

The function $\frac{1}{(1-x)^2}$ assumes only positive values (Fig. 34).

Example 2. We shall prove that $\lim\limits_{x \to 0}\left(-\frac{1}{x}\right) = \infty$. Indeed, for any $M > 0$ we have

$$\left|-\frac{1}{x}\right| > M$$

provided

$$|x| = |x-0| < \frac{1}{M} = \delta$$

Here $\left(-\frac{1}{x}\right) > 0$ for $x < 0$ and $\left(-\frac{1}{x}\right) < 0$ for $x > 0$ (Fig. 35).

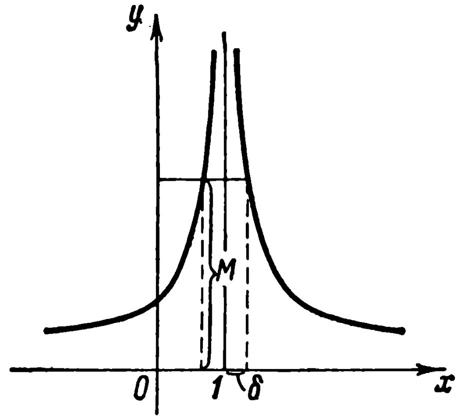

Fig. 34

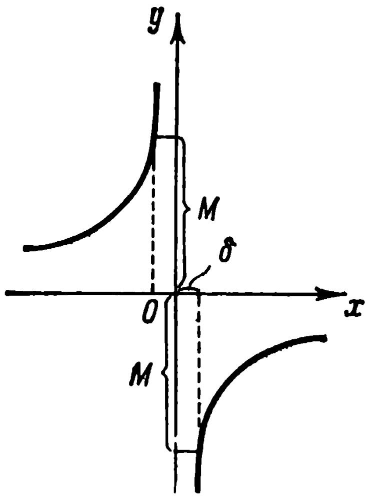

Fig. 35

If the function $f(x)$ approaches infinity as $x \to \infty$, we write

$$\lim_{x \to \infty} f(x) = \infty$$

and we may have the particular cases

$$\lim_{x \to +\infty} f(x) = \infty, \quad \lim_{x \to -\infty} f(x) = \infty, \quad \lim_{x \to +\infty} f(x) = -\infty$$

For example,

$$\lim_{x \to \infty} x^2 = +\infty, \quad \lim_{x \to -\infty} x^3 = -\infty \text{ and the like.}$$

Note 1. The function $y = f(x)$ may not approach a finite limit or infinity as $x \to a$ or as $x \to \infty$.

Example 3. The function $y = \sin x$ defined on the infinite interval $-\infty < x < +\infty$, does not approach either a finite limit or infinity as $x \to +\infty$ (Fig. 36).

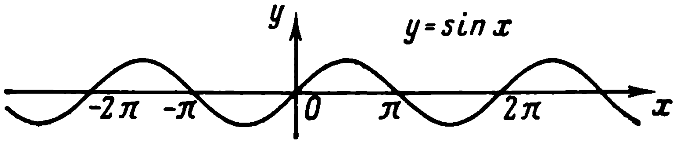

Fig. 36

Example 4. The function $y = \sin \frac{1}{x}$ defined for all values of x, except $x = 0$, does not approach either a finite limit or infinity as $x \to 0$. The graph of this function is shown in Fig. 37.

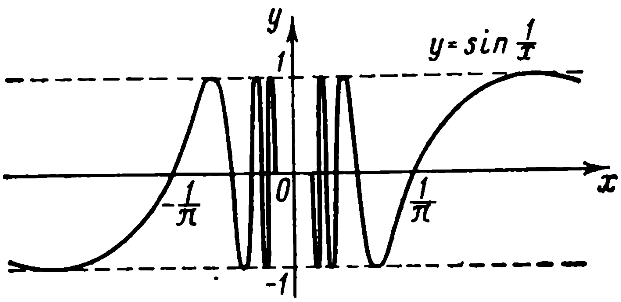

Fig. 37

Definition 2. A function $y = f(x)$ is called *bounded* in a given range of the argument x if there exists a positive number M such that for all values of x in the range under consideration the inequality $|f(x)| \leqslant M$ is fulfilled. If there is no such number M, the function $f(x)$ is called *unbounded* in the given range.

Example 5. The function $y = \sin x$, defined in the infinite interval $-\infty < x < +\infty$, is bounded, since for all values of x

$$|\sin x| \leqslant 1 = M$$

Definition 3. The function $f(x)$ is called *bounded as* $x \to a$ if there exists a neighbourhood, with centre at the point a, in which the given function is bounded.

Definition 4. The function $y = f(x)$ is called *bounded as* $x \to \infty$ if there exists a number $N > 0$ such that for all values of x satisfying the inequality $|x| > N$, the function $f(x)$ is bounded.

The boundedness of a function approaching a limit is decided by the following theorem.

Theorem 1. *If* $\lim\limits_{x \to a} f(x) = b$, *where* b *is a finite number, the function* $f(x)$ *is bounded as* $x \to a$.

Proof. From the equation $\lim\limits_{x \to a} f(x) = b$ it follows that for any $\varepsilon > 0$ there will be a δ such that in the neighbourhood $a - \delta <$

$< x < a+\delta$ the inequality

$$|f(x)-b| < \varepsilon$$

or

$$|f(x)| < |b|+\varepsilon$$

is fulfilled, which means that the function $f(x)$ is bounded as $x \to a$.

Note 2. From the definition of a bounded function $f(x)$ it follows that if

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to \infty} f(x) = \infty$$

that is, if $f(x)$ is an infinitely large function, it is unbounded. The converse is not true: an unbounded function may not be infinitely large.

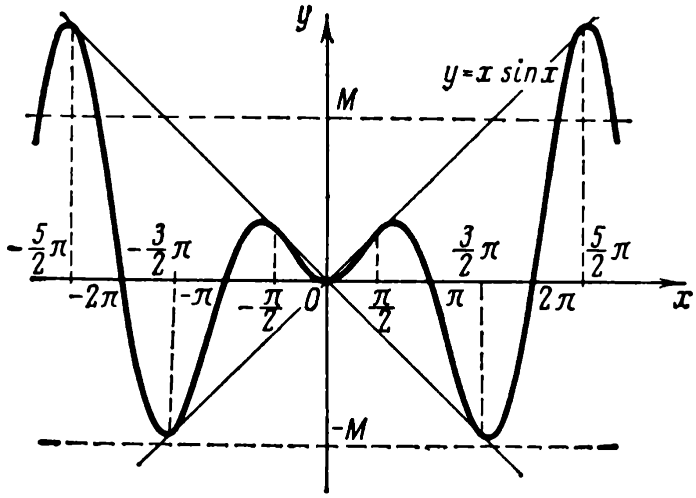

Fig. 38

For example, the function $y = x\sin x$ as $x \to \infty$ is unbounded because, for any $M > 0$, values of x can be found such that $|x\sin x| > M$. But the function $y = x\sin x$ is not infinitely large because it becomes zero when $x = 0, \pi, 2\pi, \ldots$. The graph of the function $y = x\sin x$ is shown in Fig. 38.

Theorem 2. *If* $\lim\limits_{x \to a} f(x) = b \neq 0$, *then the function* $y = \frac{1}{f(x)}$ *is a bounded function as* $x \to a$.

Proof. From the statement of the theorem it follows that for an arbitrary $\varepsilon > 0$ in a certain neighbourhood of the point $x = a$ we will have $|f(x)-b| < \varepsilon$, or $||f(x)|-|b|| < \varepsilon$, or $-\varepsilon < |f(x)| - |b| < \varepsilon$, or $|b|-\varepsilon < |f(x)| < |b|+\varepsilon$.

From the latter inequality it follows that

$$\frac{1}{|b|-\varepsilon} > \frac{1}{|f(x)|} > \frac{1}{|b|+\varepsilon}$$

For example, taking $\varepsilon = \frac{1}{10}|b|$, we get

$$\frac{10}{9|b|} > \frac{1}{|f(x)|} > \frac{10}{11|b|}$$

which means that the function $\frac{1}{f(x)}$ is bounded.

2.4 INFINITESIMALS AND THEIR BASIC PROPERTIES

In this section we shall consider functions approaching zero as the argument varies in a certain manner.

Definition. The function $\alpha = \alpha(x)$ is called *infinitesimal* as $x \to a$ or as $x \to \infty$ if $\lim\limits_{x \to a} \alpha(x) = 0$ or $\lim\limits_{x \to \infty} \alpha(x) = 0$.

From the definition of a limit it follows that if, for example, $\lim\limits_{x \to a} \alpha(x) = 0$, this means that for any preassigned arbitrarily small positive ε there will be a $\delta > 0$ such that for all x satisfying the condition $|x-a| < \delta$, the condition $|\alpha(x)| < \varepsilon$ will be satisfied.

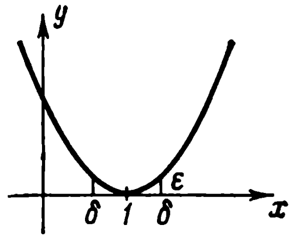

Fig. 39

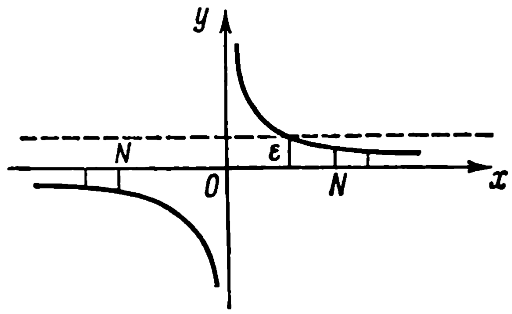

Fig. 40

Example 1. The function $\alpha = (x-1)^2$ is an infinitesimal as $x \to 1$ because $\lim\limits_{x \to 1} \alpha = \lim\limits_{x \to 1} (x-1)^2 = 0$ (Fig. 39).

Example 2. The function $\alpha = \frac{1}{x}$ is an infinitesimal as $x \to \infty$ (Fig. 40) (see Example 3, Sec. 2.2).

Let us establish a relationship that will be important later on.

Theorem 1. *If the function $y = f(x)$ is in the form of a sum of a constant b and an infinitesimal α:*

$$y = b + \alpha \tag{1}$$

then

$$\lim y = b \quad (as\ x \to a\ or\ x \to \infty)$$

Conversely, *if* $\lim y = b$, *we may write* $y = b + \alpha$, *where* α *is an infinitesimal.*

Proof. From (1) it follows that $|y-b| = |\alpha|$. But for an arbitrary ε, all values of α, from a certain value onwards, satisfy the relationship $|\alpha| < \varepsilon$; consequently, the inequality $|y-b| < \varepsilon$ will be fulfilled for all values of y from a certain value onwards. And this means that $\lim y = b$.

Conversely: if $\lim y = b$, then, given an arbitrary ε, for all values of y from a certain value onwards, we will have $|y-b| < \varepsilon$. But if we denote $y-b=\alpha$, then it follows that for all values of α, from a certain one onwards, we will have $|\alpha| < \varepsilon$; and this means that α is an infinitesimal.

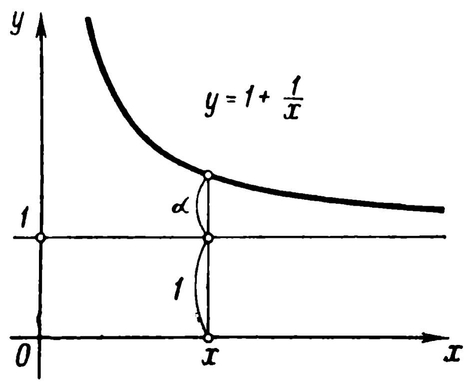

Fig. 41

Example 3. We have the function (Fig. 41)

$$y = 1 + \frac{1}{x}$$

Then

$$\lim_{x \to \infty} y = 1$$

and, conversely, since

$$\lim_{x \to \infty} y = 1$$

the variable y may be represented in the form of a sum of the limit 1 and an infinitesimal α, which in this case is $\frac{1}{x}$; that is,

$$y = 1 + \alpha$$

Theorem 2. *If $\alpha = \alpha(x)$ approaches zero as $x \to a$ (or as $x \to \infty$) and does not become zero, then $y = \frac{1}{\alpha}$ approaches infinity.*

Proof. For any $M > 0$, no matter how large, the inequality $\frac{1}{|\alpha|} > M$ will be fulfilled provided the inequality $|\alpha| < \frac{1}{M}$ is fulfilled. The latter inequality will be fulfilled for all values of α, from a certain one onwards, since $\alpha(x) \to 0$.

Theorem 3. *The algebraic sum of two, three or, in general, a definite number of infinitesimals is an infinitesimal function.*

Proof. We shall prove the theorem for two terms, since the proof is similar for any number of terms.

Let $u(x) = \alpha(x) + \beta(x)$, where $\lim_{x \to a} \alpha(x) = 0$, $\lim_{x \to a} \beta(x) = 0$. We shall prove that for any $\varepsilon > 0$, no matter how small, there will be a $\delta > 0$ such that when the inequality $|x-a| < \delta$ is satisfied, the inequality $|u| < \varepsilon$ will be fulfilled. Since $\alpha(x)$ is an infinitesimal, a δ_1 will be found such that in a neighbourhood with centre at the point a and radius δ_1 we will have

$$|\alpha(x)| < \frac{\varepsilon}{2}$$

Since $\beta(x)$ is an infinitesimal, there will be a δ_2 such that in a neighbourhood with centre at the point a and radius δ_2 we will have $|\beta(x)| < \frac{\varepsilon}{2}$.

Let us take δ equal to the smaller of the two quantities δ_1 and δ_2; then the inequalities $|\alpha| < \frac{\varepsilon}{2}$ and $|\beta| < \frac{\varepsilon}{2}$ will be fulfilled in a neighbourhood of the point a of radius δ. Hence, in this neighbourhood we will have

$$|u| = |\alpha(x) + \beta(x)| \leqslant |\alpha(x)| + |\beta(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and so $|u| < \varepsilon$, as required.

The proof is similar for the case when

$$\lim_{x \to \infty} \alpha(x) = 0, \quad \lim_{x \to \infty} \beta(x) = 0$$

Note. Later on we will have to consider sums of infinitesimals such that the number of terms increases with a decrease in each term. In this case, the theorem may not hold. To take an example, consider $u = \underbrace{\frac{1}{x} + \frac{1}{x} + \ldots + \frac{1}{x}}_{x \text{ terms}}$ where x takes on only positive integral values ($x = 1, 2, 3, \ldots, n, \ldots$). It is obvious that as $x \longrightarrow \infty$ each term is an infinitesimal, but the sum $u = 1$ is not an infinitesimal.

Theorem 4. *The product of the function of an infinitesimal $\alpha = \alpha(x)$ by a bounded function $z = z(x)$, as $x \longrightarrow a$ (or $x \longrightarrow \infty$) is an infinitesimal quantity (function).*

Proof. Let us prove the theorem for the case $x \longrightarrow a$. For a certain $M > 0$ there will be a neighbourhood of the point $x = a$ in which the inequality $|z| < M$ will be satisfied. For any $\varepsilon > 0$ there will be a neighbourhood in which the inequality $|\alpha| < \frac{\varepsilon}{M}$ will be fulfilled. The following inequality will be fulfilled in the least of these two neighbourhoods:

$$|\alpha z| < \frac{\varepsilon}{M} M = \varepsilon$$

which means that αz is an infinitesimal. The proof is similar for the case $x \longrightarrow \infty$. Two corollaries follow from this theorem.

Corollary 1. *If* $\lim \alpha = 0$, $\lim \beta = 0$, *then* $\lim \alpha\beta = 0$ because $\beta(x)$ is a bounded quantity. This holds for any finite number of factors.

Corollary 2. *If* $\lim \alpha = 0$ *and* $c = \text{const}$, *then* $\lim c\alpha = 0$.

Theorem 5. *The quotient $\frac{\alpha(x)}{z(x)}$ obtained by dividing the infinitesimal $\alpha(x)$ by a function whose limit differs from zero is an infinitesimal.*

Proof. Let $\lim \alpha(x)=0$, $\lim z(x)=b\neq 0$. By Theorem 2, Sec. 2.3, it follows that $\frac{1}{z(\alpha)}$ is a bounded quantity. For this reason, the fraction $\frac{\alpha(x)}{z(x)}=\alpha(x)\frac{1}{z(x)}$ is a product of an infinitesimal by a bounded quantity, that is, an infinitesimal.

2.5 BASIC THEOREMS ON LIMITS

In this section, as in the preceding one, we shall consider sets of functions that depend on the same argument x, where $x\longrightarrow a$ or $x\longrightarrow\infty$.

We shall carry out the proof for one of these cases, since the other is proved analogously. Sometimes we will not even write $x\longrightarrow a$ or $x\longrightarrow\infty$, but will take one or the other of them for granted.

Theorem 1. *The limit of an algebraic sum of two, three or, in general, any definite number of variables is equal to the algebraic sum of the limits of these variables:*

$$\lim(u_1+u_2+\ldots+u_k)=\lim u_1+\lim u_2+\ldots+\lim u_k$$

Proof. We shall carry out the proof for two terms since it is the same for any number of terms. Let $\lim u_1=a_1$, $\lim u_2=a_2$. Then on the basis of Theorem 1, Sec. 2.4, we can write

$$u_1=a_1+\alpha_1,\quad u_2=a_2+\alpha_2,$$

where α_1 and α_2 are infinitesimals. Consequently,

$$u_1+u_2=(a_1+a_2)+(\alpha_1+\alpha_2)$$

Since (a_1+a_2) is a constant and $(\alpha_1+\alpha_2)$ is an infinitesimal, again by Theorem 1, Sec. 2.4, we conclude that

$$\lim(u_1+u_2)=a_1+a_2=\lim u_1+\lim u_2$$

Example 1.

$$\lim_{x\to\infty}\frac{x^2+2x}{x^2}=\lim_{x\to\infty}\left(1+\frac{2}{x}\right)=\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{2}{x}=1+\lim_{x\to\infty}\frac{2}{x}=1+0=1$$

Theorem 2. *The limit of a product of two, three or, in general, any definite number of variables is equal to the product of the limits of these variables:*

$$\lim u_1\cdot u_2\cdot\ldots\cdot u_k=\lim u_1\cdot\lim u_2\cdot\ldots\cdot\lim u_k$$

Proof. To save space we carry out the proof for two factors. Let $\lim u_1=a_1$, $\lim u_2=a_2$. Therefore,

$$u_1=a_1+\alpha_1,\quad u_2=a_2+\alpha_2,$$
$$u_1u_2=(a_1+\alpha_1)(a_2+\alpha_2)=a_1a_2+a_1\alpha_2+a_2\alpha_1+\alpha_1\alpha_2$$

The product a_1a_2 is a constant. By the theorems of Sec. 2.4, the quantity $a_1\alpha_2+a_2\alpha_1+\alpha_1\alpha_2$ is an infinitesimal. Hence, $\lim u_1u_2 = a_1a_2=\lim u_1\cdot\lim u_2$.

Corollary. *A constant factor may be taken outside the limit sign.*

Indeed, if $\lim u_1=a_1$, c is a constant and, consequently, $\lim c=c$, then $\lim (cu_1)=\lim c\cdot\lim u_1 = c\cdot\lim u_1$, as required.

Example 2.

$$\lim_{x\to 2} 5x^3 = 5\lim_{x\to 2} x^3 = 5\cdot 8 = 40$$

Theorem 3. *The limit of a quotient of two variables is equal to the quotient of the limits of these variables if the limit of the denominator is not zero:*

$$\lim\frac{u}{v}=\frac{\lim u}{\lim v} \quad \textit{if} \lim v\neq 0$$

Proof. Let $\lim u=a$, $\lim v=b\neq 0$. Then $u=a+\alpha$, $v=b+\beta$, where α and β are infinitesimals.

We write the identities

$$\frac{u}{v}=\frac{a+\alpha}{b+\beta}=\frac{a}{b}+\left(\frac{a+\alpha}{b+\beta}-\frac{a}{b}\right)=\frac{a}{b}+\frac{\alpha b-\beta a}{b(b+\beta)}$$

or

$$\frac{u}{v}=\frac{a}{b}+\frac{\alpha b-\beta a}{b(b+\beta)}$$

The fraction $\frac{a}{b}$ is a constant number, while the fraction $\frac{\alpha b-\beta a}{b(b+\beta)}$ is an infinitesimal variable by virtue of Theorems 4 and 5 (Sec. 2.4), since $\alpha b-\beta a$ is an infinitesimal, while the denominator $b(b+\beta)$ has the limit $b^2\neq 0$. Thus, $\lim\frac{u}{v}=\frac{a}{b}=\frac{\lim u}{\lim v}$.

Example 3.

$$\lim_{x\to 1}\frac{3x+5}{4x-2}=\frac{\lim\limits_{x\to 1}(3x+5)}{\lim\limits_{x\to 1}(4x-2)}=\frac{3\lim\limits_{x\to 1}x+5}{4\lim\limits_{x\to 1}x-2}=\frac{3\cdot 1+5}{4\cdot 1-2}=\frac{8}{2}=4$$

Here, we made use of the already proved theorem for the limit of a fraction because the limit of the denominator differs from zero as $x\to 1$. If the limit of the denominator is zero, the theorem for the limit of a fraction is not applicable, and special considerations have to be invoked.

Example 4. Find $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$.

Here the denominator and numerator approach zero as $x\to 2$, and, consequently, Theorem 3 is inapplicable. Perform the following identical transformation:

$$\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}=x+2$$

The transformation holds for all values of x different from 2. And so, having in view the definition of a limit, we can write

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$$

Example 5. Find $\lim\limits_{x\to 1}\frac{x}{x-1}$. As $x\to 1$ the denominator approaches zero but the numerator does not (it approaches unity). Thus, the limit of the reciprocal is zero:

$$\lim_{x\to 1}\frac{x-1}{x}=\frac{\lim\limits_{x\to 1}(x-1)}{\lim\limits_{x\to 1}x}=\frac{0}{1}=0$$

Whence, by Theorem 2 of the preceding section, we have

$$\lim_{x\to 1}\frac{x}{x-1}=\infty$$

Theorem 4. *If the inequalities $u\leqslant z\leqslant v$ are fulfilled between the corresponding values of three functions $u=u(x)$, $z=z(x)$ and $v=v(x)$, where $u(x)$ and $v(x)$ approach one and the same limit b as $x\longrightarrow a$ (or as $x\longrightarrow\infty$), then $z=z(x)$ approaches the same limit as $x\longrightarrow a$ (or as $x\longrightarrow\infty$).*

Proof. For definiteness we shall consider variations of the functions as $x\longrightarrow a$. From the inequalities $u\leqslant z\leqslant v$ follow the inequalities

$$u-b\leqslant z-b\leqslant v-b$$

it is given that

$$\lim_{x\to a}u=b,\qquad \lim_{x\to a}v=b$$

Consequently, for $\varepsilon>0$ there will be a certain neighbourhood, with centre at the point a, in which the inequality $|u-b|<\varepsilon$ will be fulfilled; likewise, there will be a certain neighbourhood with centre at the point a in which the inequality $|v-b|<\varepsilon$ will be fulfilled. The following inequalities will be fulfilled in the smaller of these neighbourhoods:

$$-\varepsilon<u-b<\varepsilon \quad\text{and}\quad -\varepsilon<v-b<\varepsilon$$

and thus the inequalities

$$-\varepsilon<z-b<\varepsilon$$

will be fulfilled; that is,

$$\lim_{x\to a}z=b$$

Theorem 5. *If as $x\longrightarrow a$ (or as $x\longrightarrow\infty$) the function y takes on nonnegative values $y\geqslant 0$ and, at the same time, approaches the limit b, then b is a nonnegative number $b\geqslant 0$.*

Proof. Assume that $b < 0$, then $|y-b| \geqslant |b|$; that is, the difference modulus $|y-b|$ is greater than the positive number $|b|$ and, hence, does not approach zero as $x \to a$. But then y does not approach b as $x \to a$; this contradicts the statement of the theorem. Thus, the assumption that $b < 0$ leads to a contradiction. Consequently, $b \geqslant 0$.

In similar fashion we can prove that if $y \leqslant 0$ then $\lim y \leqslant 0$.

Theorem 6. *If the inequality $v \geqslant u$ holds between corresponding values of two functions $u = u(x)$ and $v = v(x)$ which approach limits as $x \to a$ (or as $x \to \infty$), then* $\lim v \geqslant \lim u$.

Proof. It is given that $v-u \geqslant 0$. Hence, by Theorem 5, $\lim(v-u) \geqslant 0$ or $\lim v - \lim u \geqslant 0$, and so $\lim v \geqslant \lim u$.

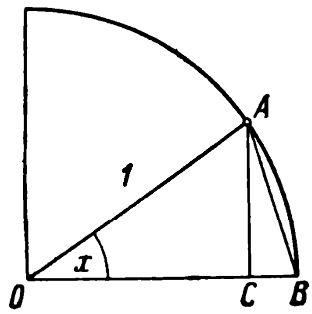

Fig. 42

Example 6. Prove that $\lim\limits_{x \to 0} \sin x = 0$.

From Fig. 42 it follows that if $OA = 1$, $x > 0$, then $AC = \sin x$, $\overset{\frown}{AB} = x$, $\sin x < x$. Obviously, when $x < 0$ we will have $|\sin x| < |x|$. By Theorems 5 and 6, it follows, from these inequalities, that $\lim\limits_{x \to 0} \sin x = 0$.

Example 7. Prove that $\lim\limits_{x \to 0} \sin \frac{x}{2} = 0$.

Indeed, $\left|\sin \frac{x}{2}\right| < |\sin x|$. Consequently, $\lim\limits_{x \to 0} \sin \frac{x}{2} = 0$.

Example 8. Prove that $\lim\limits_{x \to 0} \cos x = 1$; note that

$$\cos x = 1 - 2\sin^2 \frac{x}{2}$$

therefore,

$$\lim_{x \to 0} \cos x = \lim_{x \to 0}\left(1 - 2\sin^2 \frac{x}{2}\right) = 1 - 2\lim_{x \to 0} \sin^2 \frac{x}{2} = 1 - 0 = 1.$$

In some investigations concerning the limits of variables, one has to solve two independent problems:

(1) to prove that the limit of the variable exists and to establish the boundaries within which the limit under consideration exists;

(2) to calculate the limit to the necessary degree of accuracy.

The first problem is sometimes solved by means of the following theorem which will be important later on.

Theorem 7. *If a variable v is an increasing variable, that is, each subsequent value is greater than the preceding one, and if it is bounded, that is, $v < M$, then this variable has the limit* $\lim v = a$, *where $a \leqslant M$.*

A similar assertion may be made with respect to a decreasing bounded variable quantity.

We do not give the proof of this theorem here since it is based on the theory of real numbers, which we do not consider in this text.*

In the following two sections we shall derive the limits of two functions that find wide application in mathematics.

2.6 THE LIMIT OF THE FUNCTION $\frac{\sin x}{x}$ AS $x \to 0$

The function $\frac{\sin x}{x}$ is not defined for $x=0$ since the numerator and denominator of the fraction become zero. Let us find the limit of this function as $x \to 0$. We consider a circle of radius 1 (Fig. 43); denote the central angle MOB by x $\left(0 < x < \frac{\pi}{2}\right)$. From Fig. 43 it follows that

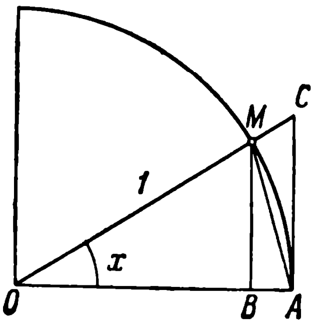

Fig. 43

$$\text{area of } \triangle MOA < \text{area of sector } MOA < \text{area of } \triangle COA. \qquad (1)$$

The area of $\triangle MOA = \frac{1}{2} OA \cdot MB = \frac{1}{2} \cdot 1 \cdot \sin x = \frac{1}{2} \sin x.$

The area of sector $MOA = \frac{1}{2} OA \cdot \overset{\frown}{AM} = \frac{1}{2} \cdot 1 \cdot x = \frac{1}{2} x.$

The area of $\triangle COA = \frac{1}{2} OA \cdot AC = \frac{1}{2} \cdot 1 \cdot \tan x = \frac{1}{2} \tan x.$

After cancelling $\frac{1}{2}$, inequalities (1) can be rewritten as

$$\sin x < x < \tan x$$

Divide all terms by $\sin x$:

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

or

$$1 > \frac{\sin x}{x} > \cos x$$

We derived this inequality on the assumption that $x > 0$; noting that $\frac{\sin(-x)}{(-x)} = \frac{\sin x}{x}$ and $\cos(-x) = \cos x$, we conclude that it holds for $x < 0$ as well. But $\lim\limits_{x \to 0} \cos x = 1$, $\lim\limits_{x \to 0} 1 = 1$.

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* The proof of this theorem is given in G. M. Fikhtengolts' *Principles of Mathematical Analysis*, Vol. I, Fizmatgiz, 1960 (in Russian).

Hence, the variable $\frac{\sin x}{x}$ lies between two quantities that have the same limit (unity). Thus by Theorem 4 of the preceding section,

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

The graph of the function $y=\frac{\sin x}{x}$ is shown in Fig. 44.

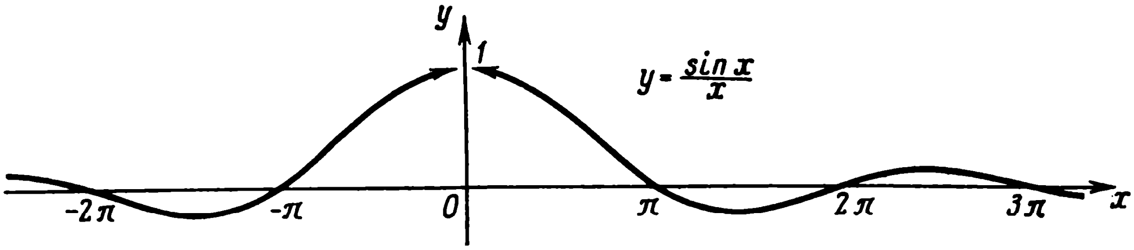

Fig. 44

Examples.

1. $\lim\limits_{x\to 0}\frac{\tan x}{x}=\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\lim\limits_{x\to 0}\frac{1}{\cos x}=1\cdot\frac{1}{1}=1.$

2. $\lim\limits_{x\to 0}\frac{\sin kx}{x}=\lim\limits_{x\to 0}k\,\frac{\sin kx}{kx}=k\lim\limits_{\substack{x\to 0\\(kx\to 0)}}\frac{\sin (kx)}{(kx)}=k\cdot 1=k \quad (k=\text{const}).$

3. $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x}=\lim\limits_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\sin\frac{x}{2}=1\cdot 0=0.$

4. $\lim\limits_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}=\lim\limits_{x\to 0}\frac{\alpha}{\beta}\cdot\frac{\frac{\sin\alpha x}{\alpha x}}{\frac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}\,\frac{\lim\limits_{x\to 0}\frac{\sin\alpha x}{\alpha x}}{\lim\limits_{x\to 0}\frac{\sin\beta x}{\beta x}}=\frac{\alpha}{\beta}\cdot\frac{1}{1}=\frac{\alpha}{\beta}$

   $(\alpha=\text{const},\quad \beta=\text{const}).$