Lecture 3

PARAMETRIC REPRESENTATION OF A FUNCTION

Given two equations:

$$\left.\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \end{aligned}\right\} \qquad (1)$$

where t assumes values that lie in the interval $[T_1, T_2]$. To each value of t there correspond values of x and y (the functions φ and ψ are assumed to be single-valued). If one regards the values of x and y as coordinates of a point in a coordinate xy-plane, then to each value of t there will correspond a definite point in the plane. And when t varies from T_1 to T_2, this point will describe a certain curve. Equations (1) are called *parametric equations* of this curve, t is the *parameter*, and *parametric* is the way the curve is represented by equations (1).

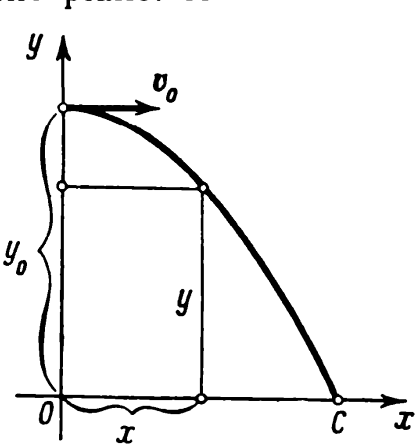

Fig. 75

Let us further assume that the function $x = \varphi(t)$ has an inverse, $t = \Phi(x)$. Then, obviously, y is a function of x;

$$y = \psi[\Phi(x)] \qquad (2)$$

Thus, equations (1) define y as a function of x, and we say that the function y of x is represented parametrically.

The explicit expression of the dependence of y on x, $y = f(x)$, is obtained by eliminating the parameter t from equations (1).

Parametric representation of curves is widely used in mechanics. If in the xy-plane there is a certain material point in motion and if we know the laws of motion of the projections of this point on the coordinate axes, then

$$\left.\begin{aligned} x &= \varphi(t) \\ y &= \psi(t) \end{aligned}\right\} \qquad (1')$$

where the parameter t is the time. Then equations (1′) are parametric equations of the trajectory of the moving point. Eliminating from these equations the parameter t, we get the equation of the trajectory in the form $y = f(x)$ or $F(x, y) = 0$. By way of illustration, let us take the following problem.

Problem. Determine the trajectory and point of impact of a load dropped from an airplane moving horizontally with a velocity v_0 at an altitude y_0 (air resistance is disregarded).

Solution. Taking a coordinate system as shown in Fig. 75, we assume that the airplane drops the load at the instant it cuts the y-axis. It is obvious that the horizontal translation of the load will be uniform and with constant velocity v_0:

$$x = v_0 t$$

Vertical displacement of the falling load due to the force of gravity will be expressed by the formula

$$s=\frac{gt^2}{2}$$

Hence the distance of the load from the ground at any instant will be

$$y=y_0-\frac{gt^2}{2}$$

The two equations

$$x=v_0t$$
$$y=y_0-\frac{gt^2}{2}$$

are the parametric equations of the trajectory. To eliminate the parameter t, we find the value $t=\frac{x}{v_0}$ from the first equation and substitute it into the second equation. Then we get the equation of the trajectory in the form

$$y=y_0-\frac{g}{2v_0^2}x^2$$

This is the equation of a parabola with vertex at the point $M(0,\ y_0)$, the y-axis serving as the axis of symmetry of the parabola.

We determine the length of OC. Denote the abscissa of C by X, and note that the ordinate of this point is $y=0$. Putting these values into the preceding formula, we get

$$0=y_0-\frac{g}{2v_0^2}X^2$$

whence

$$X=v_0\sqrt{\frac{2y_0}{g}}$$

THE EQUATIONS OF SOME CURVES IN PARAMETRIC FORM

Circle. Given a circle with centre at the coordinate origin and with radius r (Fig. 76).

Denote by t the angle formed by the x-axis and the radius to some point $M(x,\ y)$ of the circle. Then the coordinates of any point on the circle will be expressed in terms of the parameter t as follows:

$$\left.\begin{aligned} x&=r\cos t,\\ y&=r\sin t,\end{aligned}\right\}\ 0\leqslant t\leqslant 2\pi$$

These are the parametric equations of the circle. If we eliminate the parameter t from these equations, we will have an equation of the circle containing only x and y. Squaring the parametric equations and adding, we get

$$x^2+y^2=r^2(\cos^2 t+\sin^2 t)$$

or

$$x^2+y^2=r^2$$

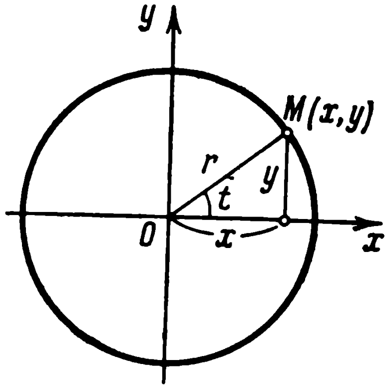

Fig. 76

Ellipse. Given the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

Set

$$x=a\cos t \tag{2'}$$

Putting this expression into equation (1) and performing the necessary manipulations, we get

$$y=b\sin t \tag{2''}$$

The equations

$$\left.\begin{array}{l} x=a\cos t, \\ y=b\sin t, \end{array}\right\} 0\leqslant t\leqslant 2\pi \tag{2}$$

are the parametric equations of the ellipse.

Let us find out the geometrical meaning of the parameter t. Draw two circles with centres at the coordinate origin and with radii a and b (Fig. 77). Let the point M (x, y) lie on the ellipse, and let B be a point of the large circle with the same abscissas as M. Denote by t the angle formed by the radius OB with the x-axis. From the figure it follows directly that

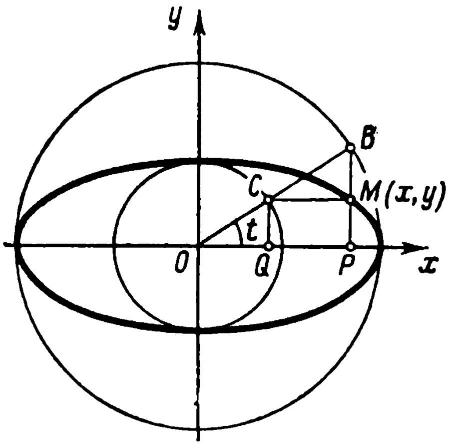

Fig. 77

$$x=OP=a\cos t \tag{2'}$$

$$CQ=b\sin t$$

From (2″) we conclude that $CQ=y$; in other words, the straight line CM is parallel to the x-axis.

Consequently, in equations (2) t is an angle formed by the radius OB and the axis of abscissas. The angle t is sometimes called an *eccentric angle*.

Cycloid. The *cycloid* is a curve described by a point lying on the circumference of a circle if the circle rolls upon a straight line without sliding (Fig. 78). Suppose that when motion began the point M of the rolling circle lay at the origin. Let us determine the coordinates of M after the circle has turned through

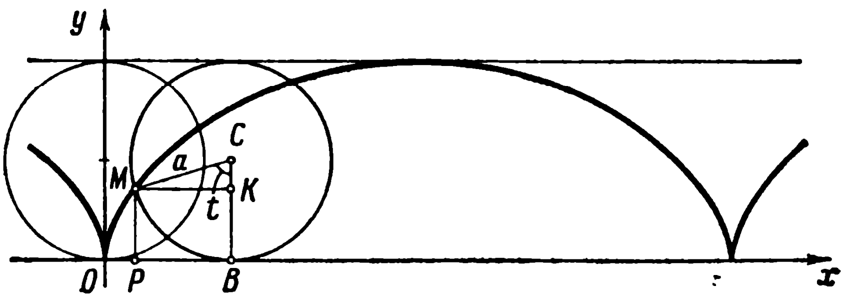

Fig. 78

an angle t. If a is the radius of the rolling circle, it will be seen from Fig. 78 that

$$x=OP=OB-PB$$

but since the circle rolls without sliding, we have

$$OB = \overset{\frown}{MB} = at, \quad PB = MK = a\sin t$$

Hence, $x = at - a\sin t = a(t - \sin t)$.

Further,

$$y = MP = KB = CB - CK = a - a\cos t = a(1 - \cos t)$$

The equations

$$\left.\begin{aligned} x &= a(t - \sin t) \\ y &= a(1 - \cos t) \end{aligned}\right\} 0 \leqslant t \leqslant 2\pi \tag{3}$$

are the parametric equations of the cycloid. As t varies between 0 and 2π, the point M will describe one arch of the cycloid.

Eliminating the parameter t from the latter equations, we get x as a function of y directly. In the interval $0 \leqslant t \leqslant \pi$, the function $y = a(1 - \cos t)$ has an inverse:

$$t = \arccos \frac{a-y}{a}$$

Substituting the expression for t into the first of equations (3), we get

$$x = a \arccos \frac{a-y}{a} - a \sin\left(\arccos \frac{a-y}{a}\right)$$

or

$$x = a \arccos \frac{a-y}{a} - \sqrt{2ay - y^2} \quad \text{when } 0 \leqslant x \leqslant \pi a$$

Examining the figure we note that when $\pi a \leqslant x \leqslant 2\pi a$

$$x = 2\pi a - \left(a \arccos \frac{a-y}{a} - \sqrt{2ay - y^2}\right)$$

It will be noted that the function

$$x = a(t - \sin t)$$

has an inverse, but it is not expressible in terms of elementary functions. And so the function $y = f(x)$ is not expressible in terms of elementary functions either.

Note 1. The cycloid clearly shows that in certain cases it is more convenient to use the parametric equations for studying functions and curves than the direct relationship of y and x (y as a function of x or x as a function of y).

Astroid. The *astroid* is a curve represented by the following parametric equations:

$$\left.\begin{aligned} x &= a\cos^3 t \\ y &= a\sin^3 t \end{aligned}\right\} 0 \leqslant t \leqslant 2\pi \tag{4}$$

Raising the terms of both equations to the power 2/3 and adding, we get the following relationship between x and y:

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} (\cos^2 t + \sin^2 t)$$

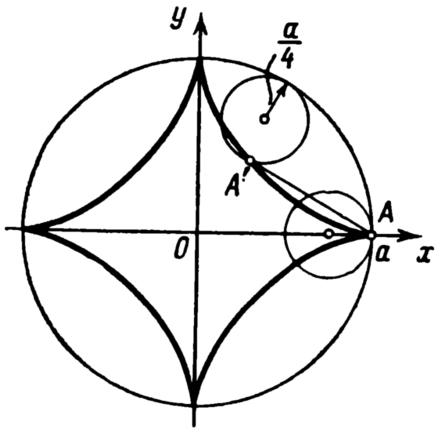

Fig. 79

or

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}} \tag{5}$$

Later on (Sec. 5.12) it will be shown that this curve is of the form shown in Fig. 79. It can be obtained as the trajectory of a certain point on the circumference of a circle of radius $a/4$ rolling (without sliding) upon another circle of radius a (the smaller circle always remains inside the larger one, see Fig. 79).

Note 2. It will be noted that equations (4) and equation (5) define more than one function $y=f(x)$. They define two continuous functions on the interval $-a \leqslant x \leqslant +a$. One takes on nonnegative values, the other nonpositive values.