LECTURE 13

THE CURVATURE OF A CURVE

ARC LENGTH AND ITS DERIVATIVE

Let the arc of a curve M_0M (Fig. 137) be the graph of a function $y=f(x)$ defined on an interval (a, b). Let us determine the arc length of the curve. On the curve M_0M take the points M_0, M_1, M_2, ..., M_{i-1}, M_i, ..., M_{n-1}, M. Connecting the points we get a broken line $M_0M_1M_2 \ldots M_{i-1}M_i \ldots M_{n-1}M$ inscribed in the arc M_0M. Denote the length of this broken line by P_n.

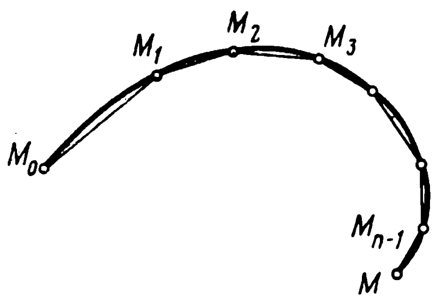

Fig. 137

The *length of the arc* M_0M is the limit (we denote it by s) approached by the length of the broken line as the largest of the lengths of the segments of the broken line $M_{i-1}M_i$ approaches zero, if this limit exists and is independent of any choice of points of the broken line $M_0M_1M_2 \ldots M_{i-1}M_i \ldots M_{n-1}M$.

It will be noted that this definition of the arc length of an arbitrary curve is similar to the definition of the length of the circumference of a circle.

In Ch. 12 it will be proved that if a function $f(x)$ and its derivative $f'(x)$ are continuous on an interval $[a, b]$, then the arc of the curve $y=f(x)$ lying between the points $[a, f(a)]$ and $[b, f(b)]$ has a definite length; a method will be shown for computing this length. It will also be established (as a corollary) that under the given conditions the ratio of the length of any arc of this curve to the length of its chord approaches unity when the length of the chord approaches zero, that is,

$$\lim_{M_0M \to 0} \frac{\text{length } \widehat{M_0M}}{\text{length } \overline{M_0M}} = 1$$

This theorem may be readily proved for the circumference* of

* Consider the arc AB, the central angle of which is 2α (Fig. 138). The length of this arc is $2R\alpha$ (R is the radius of the circle), and the length of its chord is $2R \sin\alpha$. Therefore, $\lim\limits_{\alpha\to 0} \frac{\text{length } \widehat{AB}}{\text{length } \overline{AB}} = \lim\limits_{\alpha\to 0} \frac{2R\alpha}{2R\sin\alpha} = 1.$

a circle; however, in the general case we shall accept it without proof (Fig. 138).

Let us consider the following question.

On a plane we have a curve given by the equation

$$y = f(x).$$

Let $M_0(x_0, y_0)$ be some fixed point of the curve and $M(x, y)$, some variable point of the curve. Denote by s the arc length M_0M (Fig. 139).

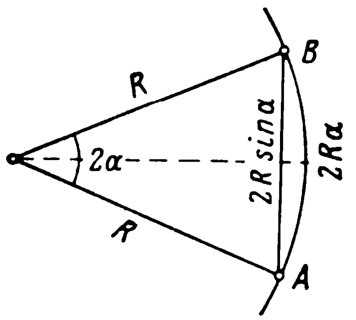

Fig. 138

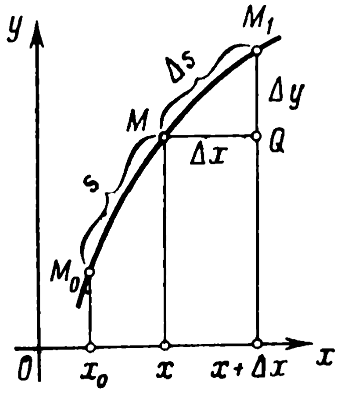

Fig. 139

The arc length s will vary with changes in the abscissa x of the point M; in other words, s is a function of x. Find the derivative of s with respect to x.

Increase x by Δx. Then the arc s will change by $\Delta s =$ the length of $\overset{\frown}{MM_1}$. Let $\overline{MM_1}$ be the chord subtending this arc. In order to find $\lim\limits_{\Delta x \to 0} \frac{\Delta s}{\Delta x}$ do as follows: from ΔMM_1Q find

$$\overline{MM_1^2} = (\Delta x)^2 + (\Delta y)^2$$

Multiply and divide the left-hand side by Δs^2:

$$\left(\frac{\overline{MM_1}}{\Delta s}\right)^2 \Delta s^2 = (\Delta x)^2 + (\Delta y)^2$$

Divide all terms of the equation by Δx^2:

$$\left(\frac{\overline{MM_1}}{\Delta s}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$

Find the limits of the left and right sides as $\Delta x \to 0$. Taking into account that $\lim\limits_{\overline{MM_1} \to 0} \frac{\overline{MM_1}}{\Delta s} = 1$ and that $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$, we get

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

or

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2} \tag{1}$$

For the *differential of the arc* we get the following expression:

$$ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \tag{2}$$

or *

$$ds=\sqrt{dx^2+dy^2} \tag{2'}$$

We have obtained an expression for the differential of arc length for the case when the curve is given by the equation $y=f(x)$. However, (2′) holds also for the case when the curve is represented by parametric equations.

If the curve is represented parametrically,

$$x=\varphi(t), \quad y=\psi(t)$$

then

$$dx=\varphi'(t)\,dt, \quad dy=\psi'(t)\,dt$$

and expression (2′) takes the form

$$ds=\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt$$

CURVATURE

One of the elements that characterize the shape of a curve is the degree of its bentness, or curvature.

Let there be a curve that does not intersect itself and has a definite tangent at each point. Draw tangents to the curve at any two points A and B and denote the angle formed by these tangents by α [or, more precisely, the angle through which the tangent turns from A to B (Fig. 140)]. This angle is called the *angle of contingence* of the arc AB. Of two arcs of the same length, that arc is more curved which has a greater angle of contingence (Figs. 140 and 141).

On the other hand, when considering arcs of different length we cannot gauge the degree of their curvature solely by the appropriate angles of contingence. Whence it follows that a complete description of the curvature of a curve is given by the **ratio** of the angle of contingence to the length of the corresponding arc.

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* Strictly speaking, (2′) holds only for the case when $dx>0$. But if $dx<0$, then $ds=-\sqrt{dx^2+dy^2}$. For this reason, in the general case this formula is more correctly written as $|ds|=\sqrt{dx^2+dy^2}$.

Definition 1. The *average curvature* K_{av} *of an arc* $\overset{\frown}{AB}$ is the ratio of the corresponding angle of contingence α to the length of the arc:

$$K_{av} = \frac{\alpha}{\overset{\frown}{AB}}$$

For one and the same curve, the average curvature of its different parts (arcs) may be different; for example, for the curve

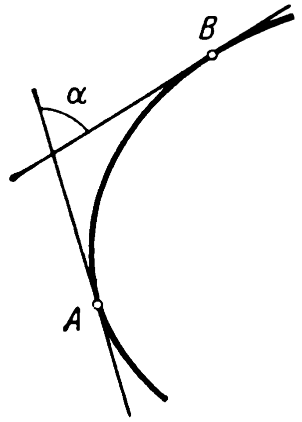

Fig. 140

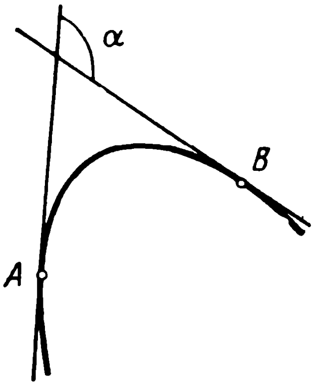

Fig. 141

shown in Fig. 142, the average curvature of the arc $\overset{\frown}{AB}$ is not equal to the average curvature of the arc $\overset{\frown}{A_1B_1}$, although the lengths of their arcs are the same. What is more, at different points the curvature of the curve differs. To characterize the degree of curvature of a given line in the immediate neighbourhood of a given point A, we introduce the concept of curvature of a curve at a given point.

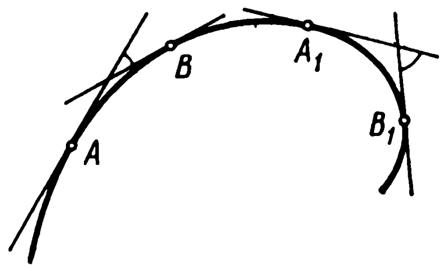

Fig. 142

Definition 2. The *curvature* K_A *of a line at a given point* A is the limit of the average curvature of the arc AB when the length of the arc approaches* zero (that is, when the point B approaches the point A):

$$K_A = \lim_{B \to A} K_{av} = \lim_{AB \to 0} \frac{\alpha}{\overset{\frown}{AB}}$$

Example. For a circle of radius r: (1) determine the average curvature of the arc $\overset{\frown}{AB}$ subtending the central angle α (Fig. 143); (2) determine the curvature at the point A.

* We assume that the magnitude of the limit does not depend on which side of the point A we take the variable point B on the curve.

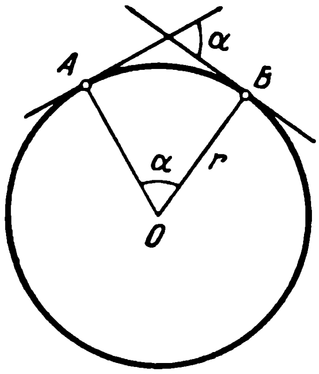

Fig. 143

Solution. (1) Obviously the angle of contingence of the arc $\overset{\frown}{AB}$ is α, the length of the arc is αr. Hence,

$$K_{av}=\frac{\alpha}{\alpha r}$$

or

$$K_{av}=\frac{1}{r}$$

(2) The curvature at the point A is

$$K=\lim_{\alpha\to 0}\frac{\alpha}{\alpha r}=\frac{1}{r}$$

Thus, the average curvature of the arc of a circle of radius r is independent of the length and position of the arc, and for all arcs it is equal to $\frac{1}{r}$. Likewise, the curvature of a circle at any point is independent of the choice of this point and is equal to $\frac{1}{r}$.

Note. It will be seen later that, generally speaking, for any curve the curvature at its various points differs.

CALCULATION OF CURVATURE

Let us develop a formula for finding the curvature of any curve at any point $M(x, y)$. We shall assume that the curve is represented in a Cartesian coordinate system by an equation of the form

$$y=f(x) \qquad (1)$$

and that the function $f(x)$ has a continuous second derivative.

Draw tangents to the curve at the points M and M_1 with abscissas x and $x+\Delta x$ and denote by φ and $\varphi+\Delta\varphi$ the angles of inclination of these tangents (Fig. 144).

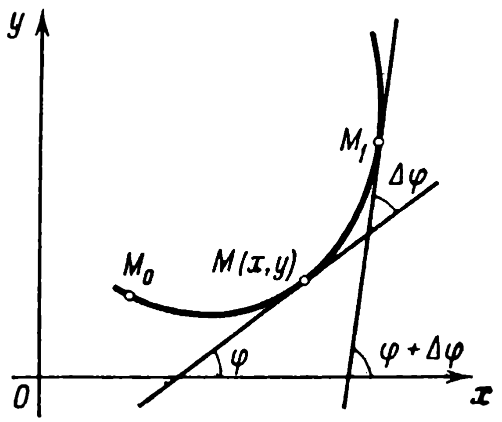

Fig. 144

We reckon the length of the arc $\overset{\frown}{M_0M}$ from some fixed point M_0 and denote it by s; then $\Delta s=\overset{\frown}{M_0M_1}-\overset{\frown}{M_0M}$, and $|\Delta s|=\overset{\frown}{MM_1}$.

As will be seen from Fig. 144, the angle of contingence corresponding to the arc $\overset{\frown}{MM_1}$ is equal to the absolute value* of the difference of the angles φ and $\varphi+\Delta\varphi$, which means it is equal to $|\Delta\varphi|$.

* It is obvious that for the curve given in Fig. 144, $|\Delta\varphi|=\Delta\varphi$ since $\Delta\varphi>0$.

According to the definition of average curvature of a curve, on the segment MM_1 we have

$$K_{av} = \frac{|\Delta\varphi|}{|\Delta s|} = \left|\frac{\Delta\varphi}{\Delta s}\right|^{\bullet}$$

To obtain the **curvature at the point** M, it is necessary to find the limit of the expression obtained on the condition that the arc length $\widehat{MM_1}$ approaches zero:

$$K = \lim_{\Delta s \to 0} \left|\frac{d\varphi}{\Delta s}\right|$$

Since the quantities φ and s both depend on x (are functions of x), φ may thus be considered as a function of s. We may consider that this function is represented parametrically by means of the parameter x. Then

$$\lim_{\Delta s \to 0} \frac{\Delta\varphi}{\Delta s} = \frac{d\varphi}{ds}$$

and, consequently,

$$K = \left|\frac{d\varphi}{ds}\right| \tag{2}$$

To calculate $\frac{d\varphi}{ds}$, we make use of the formula for differentiating a function represented parametrically:

$$\frac{d\varphi}{ds} = \frac{\frac{d\varphi}{dx}}{\frac{ds}{dx}}$$

To express the derivative $\frac{d\varphi}{dx}$ in terms of the function $y = f(x)$, we note that $\tan\varphi = \frac{dy}{dx}$ and, therefore,

$$\varphi = \arctan\frac{dy}{dx}$$

Differentiating this equation with respect to x, we get

$$\frac{d\varphi}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}$$

As for the derivative $\frac{ds}{dx}$, we found in Sec. 6.1 that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Therefore,

$$\frac{d\varphi}{ds}=\frac{\frac{d\varphi}{dx}}{\frac{ds}{dx}}=\frac{\frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=\frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

or, since $K=\left|\frac{d\varphi}{ds}\right|$, we finally get

$$K=\frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \tag{3}$$

It is thus possible to find the curvature at any point of a curve where a second derivative $\frac{d^2y}{dx^2}$ exists and is continuous. Calculations are done with formula (3). It should be noted that when calculating the curvature of a curve only the positive value of the root in the denominator should be taken, since the curvature of a line cannot (by definition) be negative.

Example 1. Determine the curvature of the parabola $y^2=2px$:
(a) at an arbitrary point $M(x, y)$;
(b) at the point $M_1(0, 0)$;
(c) at the point $M_2\left(\frac{p}{2}, p\right)$.

Solution. Find the first and second derivatives of the function $y=\sqrt{2px}$:

$$\frac{dy}{dx}=\frac{p}{\sqrt{2px}}; \quad \frac{d^2y}{dx^2}=-\frac{p^2}{(2px)^{3/2}}$$

Substituting the expressions obtained into (3), we get

(a) $K=\frac{p^2}{(2px+p^2)^{3/2}}$

(b) $K_{\substack{x=0\\y=0}}=\frac{1}{p}$

(c) $K_{\substack{x=\frac{p}{2}\\y=p}}=\frac{1}{2\sqrt{2p}}$

Example 2. Determine the curvature of the straight line $y=ax+b$ at an arbitrary point (x, y).

Solution.

$$y'=a, \quad y''=0$$

Referring to (3) we get

$$K=0$$

Thus, a straight line is a "line of zero curvature". This very same result is readily obtainable directly from the definition of curvature.

CALCULATING THE CURVATURE OF A CURVE REPRESENTED PARAMETRICALLY

Let a curve be represented parametrically:

$$x=\varphi(t), \quad y=\psi(t)$$

Then (see Sec. 3.24):

$$\frac{dy}{dx}=\frac{\psi'(t)}{\varphi'(t)}, \quad \frac{d^2y}{dx^2}=\frac{\psi''\varphi'-\psi'\varphi''}{(\varphi')^3}$$

Substituting the expressions obtained into formula (3) of the preceding section, we get

$$K=\frac{|\psi''\varphi'-\psi'\varphi''|}{[\varphi'^2+\psi'^2]^{3/2}} \tag{1}$$

Example. Determine the curvature of the cycloid

$$x=a(t-\sin t), \quad y=a(1-\cos t)$$

at an arbitrary point $(x\ \ y)$.

Solution.

$$\frac{dx}{dt}=a(1-\cos t), \quad \frac{d^2x}{dt^2}=a\sin t, \quad \frac{dy}{dt}=a\sin t, \quad \frac{d^2y}{dt^2}=a\cos t$$

Substituting the expressions obtained into (3), we get

$$K=\frac{|a(1-\cos t)a\cos t-a\sin t\cdot a\sin t|}{[a^2(1-\cos t)^2+a^2\sin^2 t]^{3/2}}=\frac{|\cos t-1|}{2^{3/2}a(1-\cos t)^{3/2}}$$

$$=\frac{1}{2^{3/2}a(1-\cos t)^{1/2}}=\frac{1}{4a\left|\sin\frac{t}{2}\right|}$$

CALCULATING THE CURVATURE OF A CURVE GIVEN BY AN EQUATION IN POLAR COORDINATES

Given a curve represented by an equation of the form

$$\rho=f(\theta) \tag{1}$$

Write the transformation formulas from polar coordinates to Cartesian coordinates:

$$\left.\begin{aligned} x&=\rho\cos\theta \\ y&=\rho\sin\theta \end{aligned}\right\} \tag{2}$$

If in these formulas we replace ρ by its expression in terms of θ, i.e., $f(\theta)$, we get

$$\left.\begin{aligned} x&=f(\theta)\cos\theta \\ y&=f(\theta)\sin\theta \end{aligned}\right\} \tag{3}$$

The latter equations may be regarded as parametric equations of curve (1), the parameter being θ.

Then

$$\frac{dx}{d\theta}=\frac{d\rho}{d\theta}\cos\theta-\rho\sin\theta,\quad \frac{dy}{d\theta}=\frac{d\rho}{d\theta}\sin\theta+\rho\cos\theta$$

$$\frac{d^2x}{d\theta^2}=\frac{d^2\rho}{d\theta^2}\cos\theta-2\frac{d\rho}{d\theta}\sin\theta-\rho\cos\theta$$

$$\frac{d^2y}{d\theta^2}=\frac{d^2\rho}{d\theta^2}\sin\theta+2\frac{d\rho}{d\theta}\cos\theta-\rho\sin\theta$$

Substituting the latter expressions into (1) of the preceding section, we get a formula for calculating the curvature of a curve in polar coordinates:

$$K=\frac{|\rho^2+2\rho'^2-\rho\rho''|}{(\rho^2+\rho'^2)^{3/2}} \quad (4)$$

Example. Determine the curvature of the spiral of Archimedes $\rho=a\theta$ $(a>0)$ at an arbitrary point (Fig. 145).

Solution.

$$\frac{d\rho}{d\theta}=a,\quad \frac{d^2\rho}{d\theta^2}=0$$

Hence

$$K=\frac{|a^2\theta^2+2a^2|}{(a^2\theta^2+a^2)^{3/2}}=\frac{1}{a}\frac{\theta^2+2}{(\theta^2+1)^{3/2}}$$

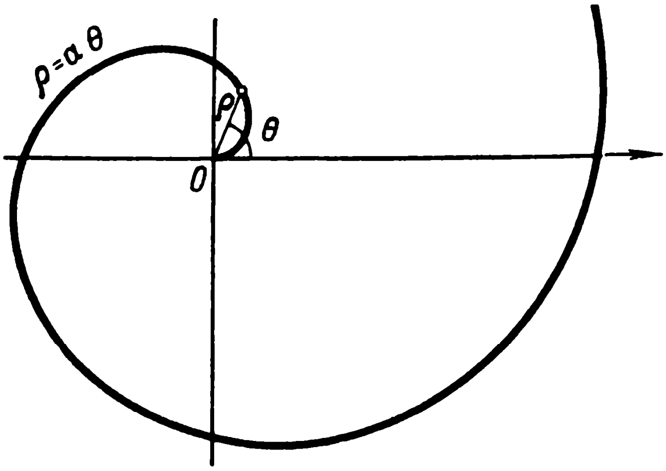

Fig. 145

It will be noted that for large values of θ we have the approximate equations $\frac{\theta^2+2}{\theta^2}\approx 1$, $\frac{\theta^2+1}{\theta^2}\approx 1$; therefore, replacing θ^2+2 by θ^2 and θ^2+1 by θ^2 in the foregoing formula, we get an approximate formula (for large values of θ):

$$K\approx\frac{1}{a}\frac{\theta^2}{(\theta^2)^{3/2}}=\frac{1}{a\theta}$$

Thus, for large values of θ the spiral of Archimedes has, approximately, the same curvature as a circle of radius $a\theta$.

THE RADIUS AND CIRCLE OF CURVATURE. THE CENTRE OF CURVATURE. EVOLUTE AND INVOLUTE

Definition. The quantity R, which is the reciprocal of the curvature K of a curve at a given point M, is called the *radius of curvature* of the curve at the point in question:

$$R=\frac{1}{K} \quad (1)$$

or

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}. \tag{2}$$

Draw a normal, at the point M, to a curve in the direction of the concavity of the curve, and lay off a segment MC equal to the radius R of the curvature of the curve at the point M. The

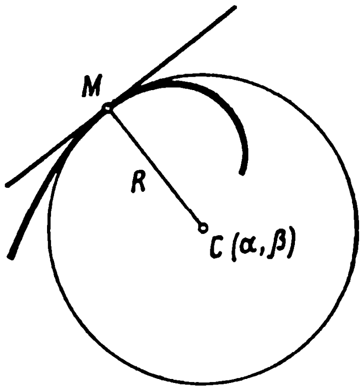

Fig. 146

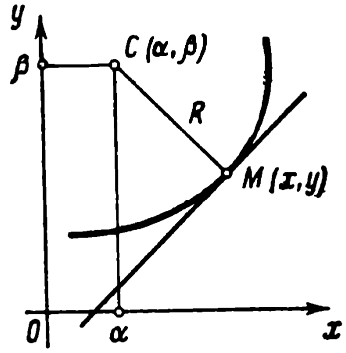

Fig. 147

point C is called the *centre of curvature* of the given curve at M; the circle, of radius R, with centre at C (passing through M) is called the *circle of curvature* of the given curve at the point M (Fig. 146).

From the definition of circle of curvature it follows that at a given point the curvature of a curve and the curvature of a circle of curvature are the same.

Let us derive formulas defining the coordinates of the centre of curvature.

Let a curve be given by the equation

$$y = f(x) \tag{3}$$

Take a point $M(x, y)$ on this curve and determine the coordinates α and β of the centre of curvature corresponding to this point (Fig. 147). To do this, write the equation of the normal to the curve at M:

$$Y - y = -\frac{1}{y'}(X - x) \tag{4}$$

(Here, X and Y are the moving coordinates of the point of the normal.)

Since the point $C(\alpha, \beta)$ lies on the normal, its coordinates must satisfy equation (4):

$$\beta - y = -\frac{1}{y'}(\alpha - x) \tag{5}$$

Further, the point $C(\alpha, \beta)$ is separated from $M(x, y)$ by a distance equal to the radius of curvature R:

$$(\alpha-x)^2+(\beta-y)^2=R^2 \tag{6}$$

Solving equations (5) and (6) simultaneously, we find α and β:

$$(\alpha-x)^2+\frac{1}{y'^2}(\alpha-x)^2=R^2$$

$$(\alpha-x)^2=\frac{y'^2}{1+y'^2}R^2$$

whence

$$\alpha=x\pm\frac{y'}{\sqrt{1+y'^2}}R, \quad \beta=y\mp\frac{1}{\sqrt{1+y'^2}}R$$

and since $R=\frac{(1+y'^2)^{3/2}}{|y''|}$, it follows that

$$\alpha=x\pm\frac{y'(1+y'^2)}{|y''|}, \quad \beta=y\mp\frac{1+y'^2}{|y''|}$$

In order to decide which signs (upper or lower) to take in the latter formulas, we must examine the case $y''>0$ and the case $y''<0$. If $y''>0$, then at this point the curve is concave, and, hence, $\beta>y$ (Fig. 147), and for this reason we take the lower signs. Taking into account that in this case $|y''|=y''$, the formulas of the coordinates of the centre of curvature will be

$$\left.\begin{aligned}\alpha&=x-\frac{y'(1+y'^2)}{y''}\\ \beta&=y+\frac{1+y'^2}{y''}\end{aligned}\right\} \tag{7}$$

Similarly, it may be shown that formulas (7) will hold for the case $y''<0$ as well.

If the curve is represented by the parametric equations

$$x=\varphi(t), \quad y=\psi(t)$$

then the coordinates of the centre of curvature are readily obtainable from (7) by substituting, in place of y' and y'', their expressions in terms of the parameter

$$y'=\frac{y'_t}{x'_t}, \quad y''=\frac{x'_t y''_t-x''_t y'_t}{x'^3_t}$$

Then

$$\left.\begin{aligned}\alpha&=x-\frac{y'(x'^2+y'^2)}{x'y''-x''y'}\\ \beta&=y+\frac{x'(x'^2+y'^2)}{x'y''-x''y'}\end{aligned}\right\} \tag{7'}$$

Example 1. To determine the coordinates of the centre of curvature of the parabola

$$y^2 = 2px$$

(a) at an arbitrary point $M(x, y)$, (b) at the point $M_0(0, 0)$, (c) at the point $M_1\left(\frac{p}{2}, p\right)$.

Solution. Substituting the values $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ into (7) we get (Fig. 148):

(a) $\alpha = 3x + p$, $\beta = -\frac{(2x)^{3/2}}{\sqrt{p}}$,

(b) at $x = 0$ we find $\alpha = p$, $\beta = 0$,

(c) at $x = \frac{p}{2}$ we have $\alpha = \frac{5p}{2}$, $\beta = -p$.

If at $M_1(x, y)$ of a given curve the curvature differs from zero, then a very definite centre of curvature $C_1(\alpha, \beta)$ corresponds to this point. The totality of all centres of curvature of the given curve forms a certain new line, called the **evolute**, with respect to the first.

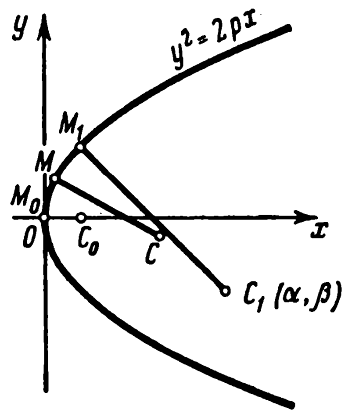

Fig. 148

Thus, the locus of centres of curvature of a given curve is called the *evolute*. As related to its evolute, the given curve is called the *evolvent* or *involute*.

If a given curve is defined by the equation $y = f(x)$, then equations (7) may be regarded as the parametric equations of the evolute with parameter x. Eliminating from these equations the parameter x (if this is possible), we get an immediate relationship between the moving coordinates of the evolute α and β. But if the curve is given by parametric equations $x = \varphi(t)$, $y = \psi(t)$, then equations (7′) yield the parametric equations of the evolute (since the quantities x, y, x', y', x'', y'' are functions of t).

Example 2. Find the equation of the evolute of the parabola

$$y^2 = 2px$$

Solution. On the basis of Example 1 we have, for any point (x, y) of the parabola,

$$\alpha = 3x + p$$

$$\beta = -\frac{(2x)^{3/2}}{\sqrt{p}}$$

Eliminating the parameter x from these equations, we get

$$\beta^2 = \frac{8}{27p}(\alpha - p)^3$$

This is the equation of a semicubical parabola (Fig. 149).

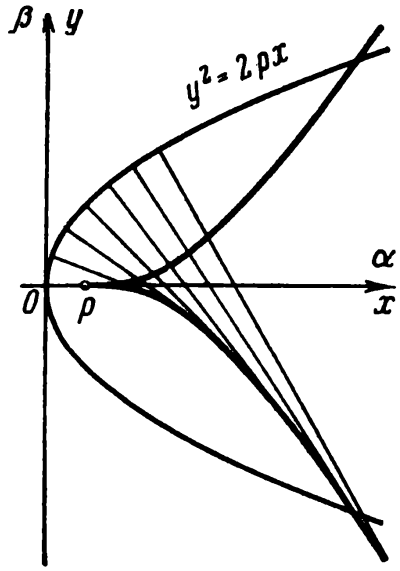

Fig. 149

Example 3. Find the equation of the evolute of an ellipse represented by the parametric equations

$$x = a\cos t, \quad y = b\sin t$$

Solution. Find the derivatives of x and y with respect to t:

$$x' = -a\sin t, \quad y' = b\cos t$$
$$x'' = -a\cos t, \quad y'' = -b\sin t$$

Substituting the expressions of the derivatives into (7′), we get

$$\alpha = a\cos t - \frac{b\cos t\,(a^2\sin^2 t + b^2\cos^2 t)}{ab\sin^2 t + ab\cos^2 t}$$
$$= a\cos t - a\cos t\sin^2 t - \frac{b^2}{a}\cos^3 t =$$
$$= \left(a - \frac{b^2}{a}\right)\cos^3 t$$

Thus,

$$\alpha = \left(a - \frac{b^2}{a}\right)\cos^3 t$$

Similarly we get

$$\beta = \left(b - \frac{a^2}{b}\right)\sin^3 t$$

Eliminating the parameter t, we get the equation of the evolute of the ellipse in the form

$$\left(\frac{\alpha}{b}\right)^{2/3} + \left(\frac{\beta}{a}\right)^{2/3} = \left(\frac{a^2-b^2}{ab}\right)^{2/3}$$

Here, α and β are the coordinates of the evolute (Fig. 150).

Example 4. Find the parametric equations of the evolute of the cycloid

$$x = a(t - \sin t)$$
$$y = a(1 - \cos t)$$

Solution.

$$x' = a(1 - \cos t), \quad y' = a\sin t$$
$$x'' = a\sin t, \quad y'' = a\cos t$$

Substituting the expressions obtained into (7′), we get

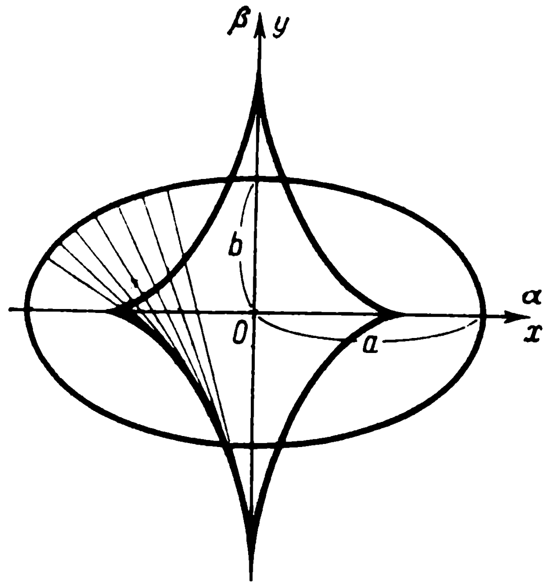

Fig. 150

$$\alpha = a(t + \sin t)$$
$$\beta = -a(1 - \cos t)$$

Make a change of variables, putting

$$\alpha = \xi - \pi a$$
$$\beta = \eta - 2a$$
$$t = \tau - \pi$$

Then the equations of the evolute will take the form

$$\xi = a(\tau - \sin\tau)$$
$$\eta = a(1 - \cos\tau)$$

They define, in coordinates ξ, η, a cycloid with the same generating circle of radius a. Thus, the evolute of a cycloid is that same cycloid displaced along the x-axis by $-\pi a$ and along the y-axis by $-2a$ (Fig. 151).

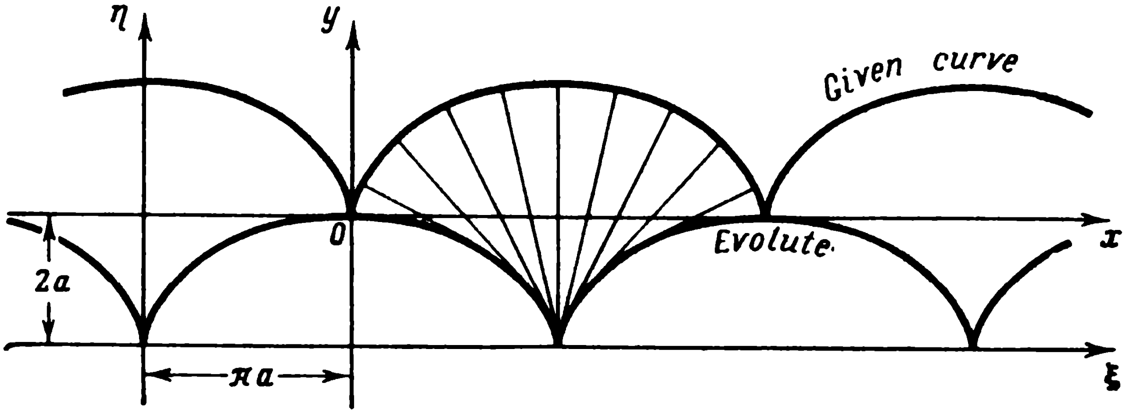

Fig. 151

THE PROPERTIES OF AN EVOLUTE

Theorem 1. *The normal to a given curve is a tangent to its evolute.*

Proof. The slope of the tangent to an evolute defined by the parametric equations (7) of the preceding section is equal to

$$\frac{d\beta}{d\alpha} = \frac{\frac{d\beta}{dx}}{\frac{d\alpha}{dx}}$$

Noting that [by virtue of the same equations (7)]

$$\frac{d\alpha}{dx} = -\frac{3y''^2y'^2 - y'y''' - y'^3y'''}{y''^2} = -y'\,\frac{3y'y''^2 - y''' - y'^2y'''}{y''^2} \qquad (1)$$

$$\frac{d\beta}{dx} = \frac{3y''^2y' - y''' - y'^2y'''}{y''^2} \qquad (2)$$

we get the relationship

$$\frac{d\beta}{d\alpha} = -\frac{1}{y'}$$

But y' is the slope of the tangent to the curve at the corresponding point; it therefore follows from the relationship obtained that the tangent to the curve and the tangent to its evolute at the corresponding point are mutually perpendicular; that is, the normal to a curve is the tangent to the evolute.

Theorem 2. *If, over a certain segment M_1M_2 of a curve, the radius of curvature varies monotonically (i.e., either only increases or only decreases), then the increment in the arc length of the evolute on this portion of the curve is equal (in absolute value) to the corresponding increment in the radius of curvature of the given curve.*

Proof. From formula (2′), Sec. 6.1, we have

$$ds^2 = d\alpha^2 + d\beta^2$$

where ds is the differential of arc length of the evolute; whence

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{d\alpha}{dx}\right)^2 + \left(\frac{d\beta}{dx}\right)^2$$

Substituting the expressions (1) and (2), we get

$$\left(\frac{ds}{dx}\right)^2 = (1+y'^2)\left(\frac{3y'y''^2 - y''' - y'^2y'''}{y''^2}\right)^2 \tag{3}$$

Then we find $\left(\frac{dR}{dx}\right)^2$. Since $R = \frac{(1+y'^2)^{3/2}}{y''}$, it follows that $R_2 = \frac{(1+y'^2)^3}{y''^2}$. Differentiating both sides of this equation with respect to x, we get the following (after appropriate manipulations):

$$2R\frac{dR}{dx} = \frac{2(1+y'^2)^2(3y'y''^2 - y''' - y'^2y''')}{(y'')^3}$$

Dividing both sides of the equation by $2R = \frac{2(1+y'^2)^{3/2}}{y''}$, we have

$$\frac{dR}{dx} = \frac{(1+y'^2)^{1/2}(3y'y''^2 - y''' - y'^2y''')}{y''^2}$$

Squaring, we get

$$\left(\frac{dR}{dx}\right)^2 = (1+y'^2)\left(\frac{3y'y''^2 - y''' - y'^2y'''}{y''^2}\right)^2 \tag{4}$$

Comparing (3) and (4), we find

$$\left(\frac{dR}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2$$

whence

$$\frac{dR}{dx} = \mp\frac{ds}{dx}$$

It is given that $\frac{dR}{dx}$ does not change sign (R only increases or only decreases); hence, $\frac{ds}{dx}$ does not change sign either. For the sake of definiteness, let $\frac{dR}{dx} \leqslant 0$, $\frac{ds}{dx} \geqslant 0$ (which corresponds to Fig. 152). Hence, $\frac{dR}{dx} = -\frac{ds}{dx}$.

Let the point M_1 have abscissa x_1 and M_2 have abscissa x_2. Apply the Cauchy theorem to the functions $s(x)$ and $R(x)$ on the interval $[x_1, x_2]$:

$$\frac{s(x_2)-s(x_1)}{R(x_2)-R(x_1)} = \frac{\left(\frac{ds}{dx}\right)_{x=\xi}}{\left(\frac{dR}{dx}\right)_{x=\xi}} = -1$$

where ξ is a number lying between x_1 and x_2 $(x_1 < \xi < x_2)$.

We introduce the designations (Fig. 152)

$$s(x_2) = s_2, \quad s(x_1) = s_1, \quad R(x_2) = R_2, \quad R(x_1) = R_1$$

Then $\frac{s_2-s_1}{R_2-R_1} = -1$ or $s_2 - s_1 = -(R_2 - R_1)$. But this means that

$$|s_2 - s_1| = |R_2 - R_1|$$

This equation is proved in exactly the same manner if the radius of curvature increases.

We have proved Theorems 1 and 2 for the case where the curve is given by an explicit function, $y = f(x)$.

If the curve is represented by parametric equations, these theorems also hold, and their proof is exactly the same.

Note. The following is a simple mechanical method for constructing an involute from its evolute.

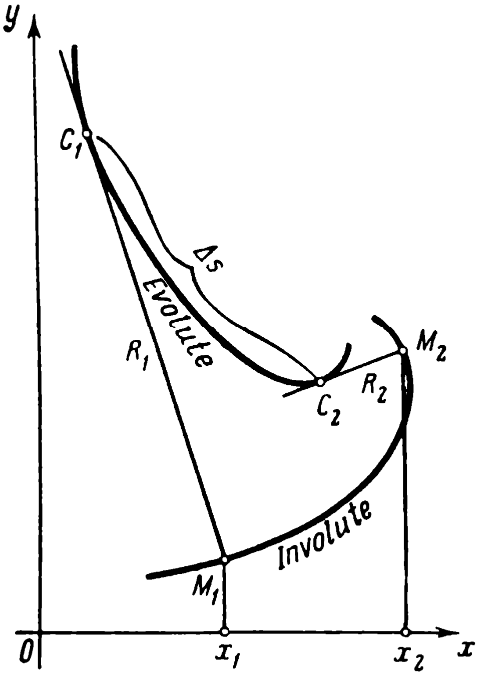

Fig. 152

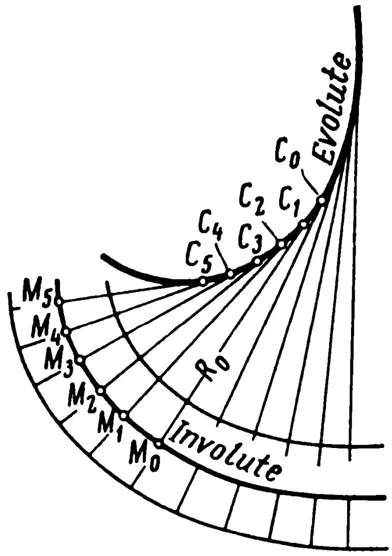

Fig. 153

Let a flexible ruler be bent into the shape of an evolute C_0C_5 (Fig. 153). Suppose one end of an unstretchable string is attached to the point C_0 and bends round the ruler. If we hold the string taut and unwind it, the end of the string will describe a curve M_5M_0, which is the involute (or evolvent, the name coming from this process of "evolving"). Proof that this curve is indeed an involute may be carried out by means of the above-established properties of the evolute.

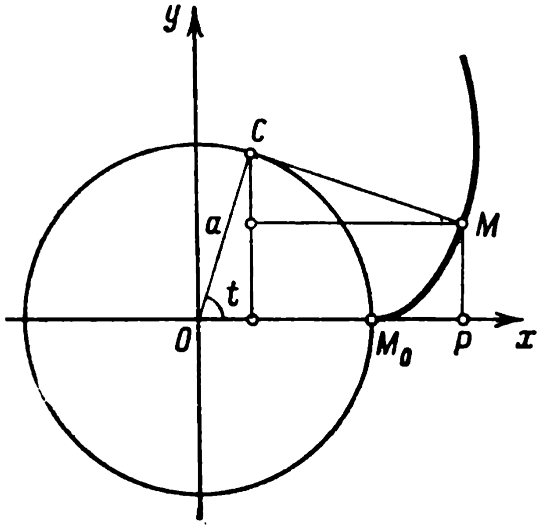

Fig. 154

It should be noted that to a single evolute there correspond an infinitude of various involutes (Fig. 153).

Example. Suppose we have a circle of radius a (Fig. 154). Take the involute of this circle that passes through the point $M_0(a, 0)$.

Taking into account that $CM = \overset{\frown}{CM_0} = at$, it is easy to obtain the equations of the involute of the circle:

$$OP = x = a(\cos t + t \sin t)$$
$$PM = y = a(\sin t - t \cos t)$$

It will be noted that the profile of a tooth of a gear wheel is most often in the shape of the involute of a circle.

APPROXIMATING THE REAL ROOTS OF AN EQUATION

Methods of investigating the behaviour of functions enable us to approximate the roots of an equation:

$$f(x) = 0$$

If the equation is an algebraic equation* of the first, second, third, or fourth degree, there are formulas which permit expressing the roots of the equation in terms of its coefficients by means of a finite number of operations of addition, subtraction, multiplication, division and evolution. Generally speaking, there are no such formulas for equations above the fourth degree. If the coefficients of any equation, algebraic or nonalgebraic (transcendental), are not literal but numerical, then the roots of the equation may be calculated approximately to any degree of accuracy. It should be noted that even when the roots of an algebraic equation are expressed

* The equation $f(x) = 0$ is called *algebraic* if $f(x)$ is a polynomial (see Sec. 7.6).

in terms of radicals, it is sometimes better to apply an approximation method of solving the equation. Below we give some methods of approximating the roots of an equation.

1. **Method of chords.** Given an equation

$$f(x) = 0 \tag{1}$$

where $f(x)$ is a continuous, doubly differentiable function on the interval $[a, b]$. Suppose that by investigating the function $y = f(x)$ within the interval $[a, b]$ we isolate a subinterval $[x_1, x_2]$ such that within this subinterval the function is monotonic (either increasing or decreasing), and at the end points the values of the function $f(x_1)$ and $f(x_2)$ have different signs. For definiteness, we say that $f(x_1) < 0$, $f(x_2) > 0$ (Fig. 155). Since the function $y = f(x)$ is continuous on the interval $[x_1, x_2]$, its graph will cut the x-axis in some one point between x_1 and x_2.

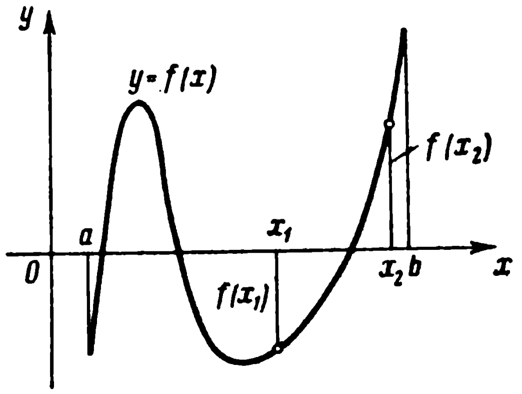

Fig. 155

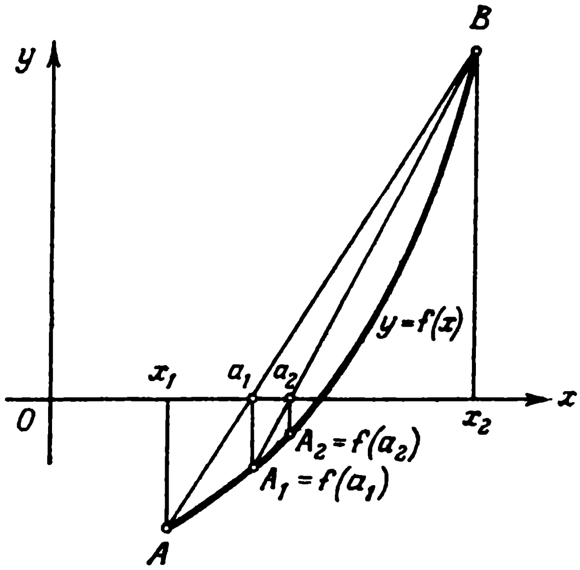

Fig. 156

Draw a chord AB connecting the end points of the curve $y = f(x)$, which correspond to abscissas x_1 and x_2. Then the abscissa a_1 of the point of intersection of this chord with the x-axis will be the approximate value of the root (Fig. 156). In order to find this approximate value let us write the equation of the straight line AB that passes through two given points $A[x_1, f(x_1)]$ and $B[x_2, f(x_2)]$:

$$\frac{y - f(x_1)}{f(x_2) - f(x_1)} = \frac{x - x_1}{x_2 - x_1}$$

Since $y = 0$ at $x = a_1$, it follows that

$$\frac{-f(x_1)}{f(x_2) - f(x_1)} = \frac{a_1 - x_1}{x_2 - x_1}$$

whence

$$a_1 = x_1 - \frac{(x_2 - x_1) f(x_1)}{f(x_2) - f(x_1)} \tag{2}$$

or

$$a_1 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \qquad (2')$$

To obtain a more exact value of the root, we determine $f(a_1)$. If $f(a_1) < 0$, then repeat the same procedure applying formula (2′) to the interval $[a_1, x_2]$. If $f(a_1) > 0$, then apply this formula to the interval $[x_1, a_1]$. By repeating this procedure several times we will obviously obtain more and more precise values of the root a_2, a_3, etc.

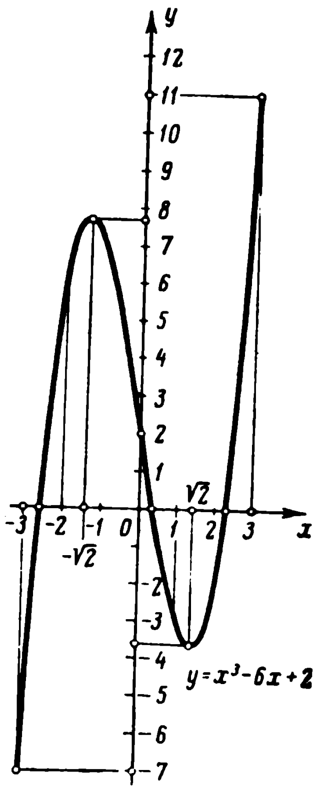

Fig. 157

Example 1. Approximate the roots of the equation

$$f(x) = x^3 - 6x + 2 = 0$$

Solution. First find the intervals where the function $f(x)$ is monotonic. Taking the derivative $f'(x) = 3x^2 - 6$, we find that it is positive for $x < -\sqrt{2}$, negative for $-\sqrt{2} < x < +\sqrt{2}$ and again positive for $x > \sqrt{2}$ (Fig. 157). Thus, the function has three intervals of monotonicity, in each of which there is one root.

To simplify the calculations, let us narrow these intervals of monotonicity (there should be a corresponding root in each interval). To do this, substitute into expression $f(x)$, at random, some values of x, then isolate (within each interval of monotonicity) shorter intervals such that the functions at the end points have different signs:

$$\left.\begin{array}{ll} x_1 = 0, & f(0) = 2 \\ x_2 = 1, & f(1) = -3 \end{array}\right\}$$
$$\left.\begin{array}{ll} x_3 = -3, & f(-3) = -7 \\ x_4 = -2, & f(-2) = 6 \end{array}\right\}$$
$$\left.\begin{array}{ll} x_5 = 2, & f(2) = -2 \\ x_6 = 3, & f(3) = 11 \end{array}\right\}$$

Thus, the roots lie within the intervals

$$(-3, -2,), \quad (0, 1), \quad (2, 3)$$

Find the approximate value of the root in the interval (0, 1); from formula (2) we have

$$a_1 = 0 - \frac{(1-0)\,2}{-3-2} = \frac{2}{5} = 0.4$$

Since

$$f(0.4) = 0.4^3 - 6 \cdot 0.4 + 2 = -0.336, \quad f(0) = 2$$

it follows that the root lies between 0 and 0.4. Again applying (2) to this interval, we get the following approximation:

$$a_2 = 0 - \frac{(0.4-0)\cdot 2}{-0.336-2} = \frac{0.8}{2.336} = 0.342, \text{ etc.}$$

We approximate the roots in the other intervals in similar fashion.

2. **Method of tangents (Newton's method).** Again, let $f(x_1) < 0$, $f(x_2) > 0$. On the interval $[x_1, x_2]$ the first derivative does not change sign. Then there is one root of the equation $f(x) = 0$ in the interval (x_1, x_2). Let us assume that the second derivative does not change sign in the interval $[x_1, x_2]$ either; this can be achieved by reducing the length of the interval within which the root lies.

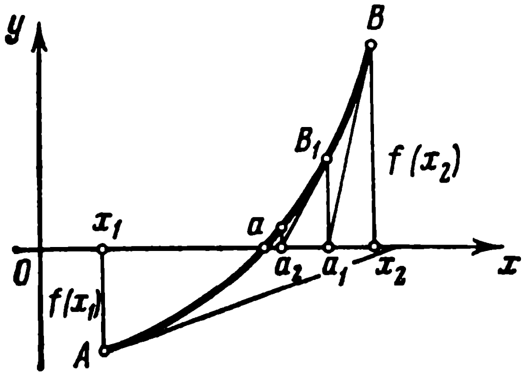

Fig. 158

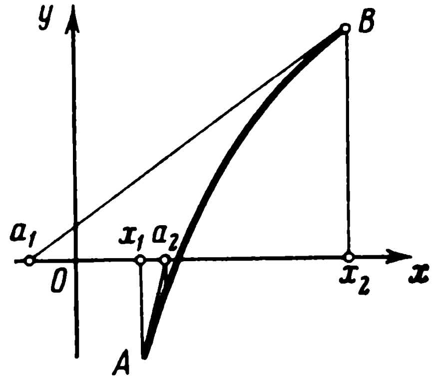

Fig. 159

Preservation of the sign of the second derivative on the interval $[x_1, x_2]$ means that the curve is either only convex or only concave on $[x_1, x_2]$.

Draw a tangent to the curve at the point B (Fig. 158). The abscissa a_1 of the point of intersection of the tangent with the x-axis will be an approximate value of the root. To find this abscissa, write the equation of the tangent at the point B:

$$y - f(x_2) = f'(x_2)(x - x_2)$$

Noting that $x = a_1$ at $y = 0$, we have

$$a_1 = x_2 - \frac{f(x_2)}{f'(x_2)} \tag{3}$$

Then, drawing the tangent line at the point $B_1[a_1, f(a_1)]$, we analogously find a more exact value of the root a_2. By repeating this procedure we can calculate the approximate value of the root to any desired degree of accuracy.

Note the following. If we drew the tangent to the curve not at the point B but at A, it might appear that the point of intersection of the tangent with the x-axis lies outside the interval (x_1, x_2).

From Figs. 158 and 159 it follows that the tangent should be drawn at the end of the arc at which the signs of the function and its second derivative coincide. Since it is given that on the interval $[x_1, x_2]$ the second derivative preserves its sign, the signs of the function and the second derivative must coincide at one of the end points. This rule also holds for the case where $f'(x) < 0$. If the tangent is drawn at the left end point of the interval, then in formula (3) we must put x_1 in place of x_2:

$$a_1 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad (3')$$

When there is a point of inflection C in the interval (x_1, x_2), the method of tangents can yield an approximate value of the root lying without the interval (x_1, x_2) (Fig. 160).

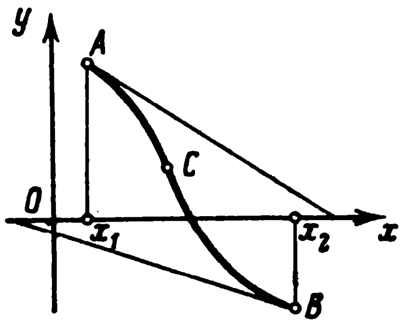

Fig. 160

Example 2. Apply formula (3′) to finding the root of the equation

$$f(x) = x^3 - 6x + 2 = 0$$

within the interval (0, 1). We have

$$f(0) = 2, \quad f'(0) = (3x^2 - 6)\,|_{x=0} = -6,$$
$$f''(x) = 6x \geqslant 0$$

and so from (3′) we get

$$a_1 = 0 - \frac{2}{-6} = \frac{1}{3} = 0.333.$$

3. **Combined method** (Fig. 161). Applying at the same time on the interval $[x_1, x_2]$ the method of chords and the method of tangents, we get two points a_1 and $\bar{a}_1$ lying on either side of the desired root a, since $f(a_1)$ and $f(\bar{a}_1)$ have different signs. Then, on the interval $[a_1, \bar{a}_1]$ again apply the method of chords and the method of tangents. This yields two numbers: a_2 and $\bar{a}_2$, which are still closer to the value of the root. We continue in this manner until the difference between the approximate values found is less than the required degree of accuracy. It will be noted that in the combined method we approach the sought-for root from two sides simultaneously (i.e., at the same time we approximate the root with an excess and with a deficit).

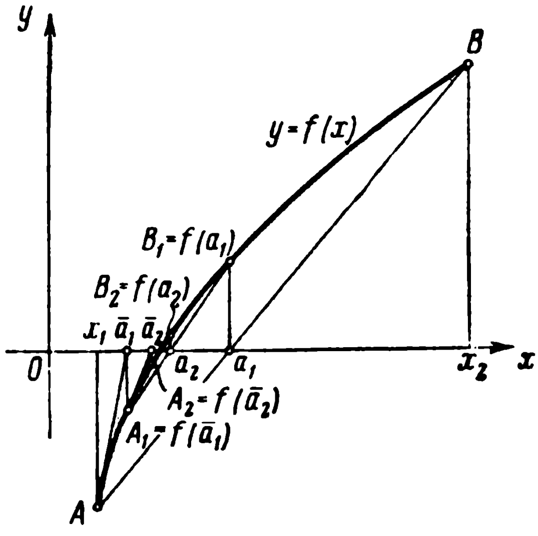

Fig. 161

To illustrate in the case we have examined it will be clear that by substitution we have

$$f(0.333) > 0, \quad f(0.342) < 0$$

Hence, the root lies between the approximate values obtained:

$$0.333 < x < 0.342$$