## INVESTIGATING THE BEHAVIOUR OF FUNCTIONS

## STATEMENT OF THE PROBLEM

A study of the quantitative aspect of natural phenomena leads (o) the establishment and study of functional relations between the variables involved. If such a functional relationship can be expressed nnalytically, that is, in the form of one or more formulas, we are llen in a position to investigate it with the tools of mathematical unalysis. For instance, a study of the flight of a shell in empty yace yields a formula that gives the dependence of the range $R$ upon the angle of elevation $\alpha$ and the initial velocity $v_{0}$ :

$$
R=\frac{v_{0}^{2} \sin 2 \alpha}{g}
$$

( $g$ is the acceleration of gravity).
With this formula we can determine at what angle $\alpha$ the range $R$ will be greatest, or least, and what the conditions must be for the range to increase as the angle $\alpha$ is increased, etc.

Let us consider another instance. Studies of oscillations of a load (in a spring (of a railway car or automobile) yielded a formula howing how the deviation $y$ of the load from a position of equilibrium depends on the time $t$ :

$$
y=e^{-k t}(A \cos \omega t+B \sin \omega t)
$$

Fior a given oscillatory system the quantities $k, A, B, \omega$ that enter into this formula have a very definite meaning (they depend upon the elasticity of the spring, the load, etc., but do not change with time $t$ ) and for this reason are considered constant.
()n the basis of this formula we can find out at what values of 1 the deviation $y$ will increase with increasing $t$, how the maximum deviation varies as a function of time, for what values of $t$ we wherve these maximum deviations, for what values of $t$ we obtain maximum velocities of motion of the load, and a number of other lhings.

All these questions are embraced by the concept "investigating the behaviour of a function". It is obviously very difficult to defermine all these questions by calculating the values of a function $" I$ specific points (like we did in Chapter 2). The purpose of this chapter is to establish more general techniques for investigating llie behaviour of functions.

## INCREASE AND DECREASE OF A FUNCTION

Theorem. (1) If a function $f(x)$, which has $a$ derivative on an interval $[a, b]$, increases on this interval, then its derivative on $[a, b]$ is not negative, that is, $f^{\prime}(x) \geqslant 0$.
(2) If the function $f(x)$ is continuous on the interval $[a, b]$ and is differentiable on $(a, b)$, where $f^{\prime}(x)>0$ for $a<x<b$, then the function increases on the interval $[a, b]$.

Proof. We start by proving the first part of the theorem. Let $f(x)$ increase on the interval [a,b]. Increase the argument $x$ by $\Delta x$ and consider the ratio

$$
\begin{equation*}
\frac{f(x+\Delta x)-f(x)}{\Delta x} \tag{1}
\end{equation*}
$$

Since $f(x)$ is an increasing function,

$$
f(x+\Delta x)>f(x) \text { for } \Delta x>0
$$

and

$$
f(x+\Delta x)<f(x) \text { for } \Delta x<0
$$

In both cases

$$
\begin{equation*}
\frac{f(x+\Delta x)-f(x)}{\Delta x}>0 \tag{2}
\end{equation*}
$$

and consequently

$$
\lim _{\Delta x \rightarrow 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \geqslant 0
$$

which means $f^{\prime}(x) \geqslant 0$, which is what we set out to prove. [If we had $f^{\prime}(x)<0$, then for sufficiently small values of $\Delta x$, ratio (1) would be negative, but this would contradict relation (2).]

Let us now prove the second part of the theorem. Let $f^{\prime}(x)>0$ for all values of $x$ on the interval $(a, b)$.

Let us consider any two values $x_{1}$ and $x_{2}, x_{1}<x_{2}$, on the interval [ $a, b$ ].

By Lagrange's mean-value theorem we have

$$
f\left(x_{2}\right)-f\left(x_{1}\right)=f^{\prime}(\xi)\left(x_{2}--x_{1}\right), \quad x_{1}<\xi<x_{2}
$$

It is given that $f^{\prime}(\xi)>0$, hence $f\left(x_{2}\right)-f\left(x_{1}\right)>0$, and this means that $f(x)$ is an increasing function.

There is a similar theorem for a decreasing (differentiable) function as well, namely:

If $f(x)$ decreases on an interval $[a, b]$, then $f^{\prime}(x) \leqslant 0$ on this interval. If $f^{\prime}(x)<0$ on $(a, b)$, then $f(x)$ decreases on $[a, b]$. [Of course, we again assume that the function is continuous at all points of $[a, b]$ and is differentiable everywhere on $(a, b)$.]

Note. The foregoing theorem expresses the following geometric lact. If on an interval $[a, b]$ a function $f(x)$ increases, then the langent to the curve $y=f(x)$ at each point on this interval forms III acute angle $\varphi$ with the $x$-axis or (at certain points) is horizontal; The tangent of this angle is not negative: $f^{\prime}(x)=\tan \varphi \geqslant 0$ (Fig. 98a). If the function $f(x)$ decreases on the interval $[a, b]$, then the angle of inclination of the tangent line forms an obtuse angle (or, at some


Fig. 98
points, the tangent line is horizontal) the tangent of this angle is not positive (Fig. 98b). We can illustrate the second part of the lheorem in similar fashion. This theorem permits judging the increase or decrease of a function by the sign of its derivative.

Example. Determine the domains of increase and decrease of the function

$$
y=x^{4}
$$

Solution. The derivative is equal to

$$
y^{\prime}=4 x^{3}
$$

Fior $x>0$ we have $y^{\prime}>0$ and the function increases; for $x<0$ we have $y^{\prime}<0$ and the function decreases (Fig. 99).

## MAXIMA AND MINIMA OF FUNCTIONS

Definition of a maximum. A function $f(x)$ has a maximum at the point $x_{1}$ if the value of the function $f(x)$ at the point $x_{1}$ is kreater than its values at all points of a certain interval containing the point $x_{1}$. In other words, the function $f(x)$ has a maximum when $x=x_{1}$ if $f\left(x_{1}+\Delta x\right)<f\left(x_{1}\right)$ for any $\Delta x$ (positive and negative) lhat are sufficiently small in absolute value.*

[^0]For example, the function $y=f(x)$, whose graph is given in Fig. 100, has a maximum at $x=x_{1}$.

Definition of a minimum. A function $f(x)$ has a minimum at $x=x_{2}$ if

$$
f\left(x_{2}+\Delta x\right)>f\left(x_{2}\right)
$$

for any $\Delta x$ (positive and negative) that are sufficiently small in absolute value (Fig. 100).

For instance, the function $y=x^{4}$ considered at the end of the preceding section (see Fig. 99) has a minimum for $x=0$, since $y=0$ when $x=0$ and $y>0$ for all other values of $x$.


Fig. 99


Fig. 100

In connection with the definitions of maximum and minimum, note the following.

1. A function defined on an interval can reach maximum and minimum values only for values of $x$ that lie within the given interval.
2. One should not think that the maximum and minimum of a function are its respective largest and smallest values over a given interval: at a point of maximum, a function has the largest value only in comparison with those values that it has at all points sufficiently close to the point of maximum, and the smallest value only in comparison with those that it has at all points sufficiently close to the minimum point.

To illustrate, take Fig. 101. Here is a function, defined on the interval [ $a, b$ ], which
at $x=x_{1}$ and $x=x_{3}$ has a maximum,
at $x=x_{2}$ and $x=x_{4}$ has a minimum,
but the minimum of the function at $x=x_{4}$ is greater than the maximum of the function at $x=x_{1}$. At $x=b$, the value of the function is greater than any maximum of the function on the interval under consideration.

The generic terms for maxima and minima of a function are rxtremum (pl. extrema) or extreme values of the function.

To some extent, the extrema of a function and their positions on the interval $[a, b]$ characterize the variation of the function versus changes in the argument.

Below we give a method for linding extrema.

Theorem 1. (A necessary condilion for the existence of an extremum). If at a point $x=x_{1} a$ differentiable function $y=f(x)$ has a maximum or minimum, its derivative vanishes at this point: $f^{\prime}\left(x_{1}\right)=0$.

Proof. For definiteness, let us assume that at the point $x=x_{1}$


Fig. 101 the function has a maximum.
Then, for sufficiently small (in absolute value) increments $\Delta x(\Delta x \neq 0)$ we have

$$
f\left(x_{1}+\Delta x\right)<f\left(x_{1}\right)
$$

that is,

$$
f\left(x_{1}+\Delta x\right)-f\left(x_{1}\right)<0
$$

But in this case the sign of the ratio

$$
\frac{f\left(x_{1}+\Delta x\right)-f\left(x_{1}\right)}{\Delta x}
$$

is determined by the sign of $\Delta x$, namely:

$$
\begin{aligned}
& \frac{f\left(x_{1}+\Delta x\right)-f\left(x_{1}\right)}{\Delta x}>0 \text { when } \Delta x<0 \\
& \frac{f\left(x_{1}+\Delta x\right)-f\left(x_{1}\right)}{\Delta x}<0 \text { when } \Delta x>0
\end{aligned}
$$

By the definition of a derivative we have

$$
f^{\prime}\left(x_{1}\right)=\lim _{\Delta x \rightarrow 0} \frac{f\left(x_{1}+\Delta x\right)-f\left(x_{1}\right)}{\Delta x}
$$

If $f\left(x_{1}\right)$ has a derivative at $x=x_{1}$, the limit on the right is independent of how $\Delta x$ approaches zero (remaining positive or negative).

But if $\Delta x \rightarrow 0$ and remains negative, then

$$
f^{\prime}\left(x_{1}\right) \geqslant 0
$$

But if $\Delta x \rightarrow 0$ and remains positive, then

$$
f^{\prime}\left(x_{1}\right) \leqslant 0
$$

Since $f^{\prime}\left(x_{1}\right)$ is a definite number that is independent of the way in which $\Delta x$ approaches zero, the latter two inequalities are consistent only if

$$
f^{\prime}\left(x_{1}\right)=0
$$

The proof is similar for the case of a minimum of a function. Corresponding to this theorem is the following obvious geometric fact: if at points of maximum and minimum, a function $f(x)$ has


Fig. 102 a derivative, the tangent line to the curve $y=f(x)$ at each point is parallel to the $x$-axis. Indeed, from the fact that $f^{\prime}\left(x_{1}\right)=\tan \varphi=0$, where $\varphi$ is the angle between the tangent line and the $x$-axis, it follows that $\varphi=0$ (Fig. 100).

From Theorem 1 it follows immediately that if for all considered values of the argument $x$ the function $f(x)$ has a derivative, then it can have an extremum (maximum or minimum) only at those values for which the derivative vanishes. The converse does not hold: it cannot be said that there definitely exists a maximum or minimum for every value at which the derivative vanishes. For instance, in Fig. 100 we have a function for which the derivative at $x=x_{3}$ vanishes (the tangent line is horizontal), yet the function at this point is neither a maximum nor a minimum.

In exactly the same way, the function $y=x^{3}$ (Fig. 102) at $x=0$ has a derivative equal to zero:

$$
\left(y^{\prime}\right)_{x=0}=\left(3 x^{2}\right)_{x=0}=0
$$

but at this point the function has neither a maximum nor a minimum. Indeed, no matter how close the point $x$ is to $O$, we will always have

$$
x^{3}<0 \text { when } x<0
$$

and

$$
x^{3}>0 \text { when } x>0
$$

We have investigated the case where a function has a derivative at all points on some closed interval. Now what about those points at which there is no derivative? The following examples will show that at these points there can only be a maximum or a minimum, but there may not be either one or the other.

Example 1. The function $y=|x|$ has no derivative at the point $x=0$ (at this point the curve does not have a definite tangent line), but the function has a minimum at this point: $y=0$ when $x=0$, whereas for any other point $x$ different from zero we have $y>0$ (Fig. 103).

Example 2. The function $y=\left(1-x^{\frac{2}{3}}\right)^{3 / 2}$ has no derivative at $x=0$, since $v^{\prime}-\left(1-x^{\frac{2}{3}}\right)^{\frac{1}{2}} x^{-\frac{1}{3}}$ becomes infinite at $x=0$, but the function has " maximum at this point: $f(0)=1, f(x)<1$ for $x$ different from zero (Fig. 104).

Example 3. The function $y=\sqrt[3]{x}$ has no derivative at $x=0\left(y^{\prime} \longrightarrow \infty\right.$ as $x \longrightarrow 0$ ). At this point the function has neither a maximum nor a minimum: $f(0)=0, f(x)<0$ for $x<0, f(x)>0$ for $x>0$ (Fig. 105).


Fig. 103


Fig. 104

Thus, a function can have an extremum only in two cases: either at points where the derivative exists and is zero, or at points where the derivative does not exist.

It must be noted that if the derivative does not exist at some point (but exists at nearby points), then at this point the derivative is discontinuous.

The values of the argument for which the derivative vanishes or is discontinuous are called critical


Fig. 105 points or critical values.

From what has been said it follows that not for every critical value does a function have a maximum or a minimum. However, if at some point the function attains a maximum or a minimum, this point is definitely critical. And so to find the extrema of a function do as follows: find all the critical points, and then, investigating separately each critical point, find out whether the function will have a maximum or a minimum at that point, or whether there will be neither maximum nor minimum.

Investigation of a function at critical points is based on the following theorem.

Theorem 2. (Sufficient conditions for the existence of an extremum). Let there be a function $f(x)$ continuous on some interval containing a critical point $x_{1}$ and differentiable at all points of the interval (with the exception, possibly, of the point $x_{1}$ itself). If in moving from left to right through this point the derivative changes sign from plus to minus, then at $x=x_{1}$ the function has a maximum.

But if in moving through the point $x_{1}$ from left to right the derivative changes sign from minus to plus, the function has a minimum at this point.

And so

$$
\text { if (a) }\left\{\begin{array}{l}
f^{\prime}(x)>0 \text { when } x<x_{1} \\
f^{\prime}(x)<0 \text { when } x>x_{1}
\end{array}\right.
$$

then at $x_{1}$ the function has a maximum;

$$
\text { if (b) }\left\{\begin{array}{l}
f^{\prime}(x)<0 \text { when } x<x_{1} \\
f^{\prime}(x)>0 \text { when } x>x_{1}
\end{array}\right.
$$

then at $x_{1}$ the function has a minimum. Note here that the conditions (a) or (b) must be fulfilled for all values of $x$ that are sufficiently close to $x_{1}$, that is, at all points of some sufficiently small neighbourhood of the critical point $x_{1}$.

Proof. Let us first assume that the derivative changes sign from plus to minus, in other words, that for all $x$ sufficiently close to $x_{1}$ we have

$$
\begin{aligned}
& f^{\prime}(x)>0 \text { when } x<x_{1} \\
& f^{\prime}(x)<0 \text { when } x>x_{1}
\end{aligned}
$$

Applying the Lagrange theorem to the difference $f(x)-f\left(x_{1}\right)$ we have

$$
f(x)-f\left(x_{1}\right)=f^{\prime}(\xi)\left(x-x_{1}\right)
$$

where $\xi$ is a point lying between $x$ and $x_{1}$.
(1) Let $x<x_{1}$; then

$$
\xi<x_{1}, f^{\prime}(\xi)>0, f^{\prime}(\xi)\left(x-x_{1}\right)<0
$$

and, consequently,

$$
f(x)-f\left(x_{1}\right)<0
$$

or

$$
\begin{equation*}
f(x)<f\left(x_{1}\right) \tag{1}
\end{equation*}
$$

(2) Let $x>x_{1}$; then

$$
\xi>x_{1}, f^{\prime}(\xi)<0, f^{\prime}(\xi)\left(x-x_{1}\right)<0
$$

and, consequently,

$$
f(x)-f\left(x_{1}\right)<0
$$

or

$$
\begin{equation*}
f(x)<f\left(x_{1}\right) \tag{2}
\end{equation*}
$$

The relations (1) and (2) show that for all values of $x$ sufficiently close to $x_{1}$ the values of the function are less than those at $x_{1}$. Hence, the function $f(x)$ has a maximum at the point $x_{1}$.

The second part of the theorem on the sufficient condition for a minimum is proved in similar fashion.

Fig. 106 illustrates the meaning of Theorem 2.
At $x=x_{1}$, suppose $f^{\prime}\left(x_{1}\right)=0$ and let the following inequalities be fulfilled for all $x$ sufficiently close to $x_{1}$ :

$$
\begin{aligned}
& f^{\prime}(x)>0 \text { when } x<x_{1} \\
& f^{\prime}(x)<0 \text { when } x>x_{1}
\end{aligned}
$$

Then for $x<x_{1}$ the tangent to the curve forms with the $x$-axis an acute angle, and the function increases, but for $x>x_{1}$ the tangent forms with the $x$-axis an obtuse angle, and the function decreases; at $x=x_{1}$ the function passes from increasing to decreasing values, which means it has a maximum.

If at $x_{2}$ we have $f^{\prime}\left(x_{2}\right)=0$ and for all values of $x$ sufficiently close to $x_{2}$ the following inequalities hold:

$$
\begin{aligned}
& f^{\prime}(x)<0 \text { when } x<x_{2} \\
& f^{\prime}(x)>0 \text { when } x>x_{2}
\end{aligned}
$$

then at $x<x_{2}$ the tangent to the


Fig. 106 curve forms with the $x$-axis an obtuse angle, the function decreases, and at $x>x_{2}$ the tangent to the curve forms an acute angle, and the function increases. At $x=x_{2}$ the function passes from decreasing to increasing values, which means it has a minimum.

If at $x=x_{3}$ we have $f^{\prime}\left(x_{3}\right)=0$ and for all values of $x$ sufficiently close to $x_{3}$ the following inequalities hold:

$$
\begin{aligned}
& f^{\prime}(x)>0 \text { when } x<x_{3} \\
& f^{\prime}(x)>0 \text { when } x>x_{3}
\end{aligned}
$$

then the function increases both for $x<x_{3}$ and for $x>x_{3}$. Therefore, at $x=x_{3}$ the function has neither a maximum nor a minimum. Such is the case with the function $y=x^{3}$ at $x=0$.

Indeed, the derivative $y^{\prime}=3 x^{2}$, hence,

$$
\begin{aligned}
& \left(y^{\prime}\right)_{x=0}=0 \\
& \left(y^{\prime}\right)_{x<0}>0 \\
& \left(y^{\prime}\right)_{x>0}>0
\end{aligned}
$$

and this means that at $x=0$ the function has neither a maximum nor a minimum (see Fig. 102).

## TESTING A DIFFERENTIABLE FUNCTION FOR MAXIMUM AND MINIMUM WITH A FIRST DERIVATIVE

The preceding section permits us to formulate a rule for testing a differentiable function, $y=f(x)$, for maximum and minimum.

1. Find the first derivative of the function, i.e., $f^{\prime}(x)$.
2. Find the critical values of the argument $x$; to do this:
(a) equate the first derivative to zero and find the real roots of the equation $f^{\prime}(x)=0$ obtained;
(b) find the values of $x$ at which the derivative $f^{\prime}(x)$ becomes discontinuous.
3. Investigate the sign of the derivative on the left and right of the critical point. Since the sign of the derivative remains constant on the interval hetween two critical points, it is sufficient, for investigating the sign of the derivative on the left and right of, say, the critical point $x_{2}$ (Fig. 106), to determine the sign of the derivative at the points $\alpha$ and $\beta\left(x_{1}<\alpha<x_{2}, x_{2}<\beta<x_{3}\right.$, where $x_{1}$ and $x_{3}$ are the closest critical points).
4. Evaluate the function $f(x)$ for every critical value of the argument.

This gives us the following diagram of possible cases:

| Signs of derivative $f^{\prime}(x)$ when passing through critical point $x_{1}$ : |  |  | Character of critical point |
| :---: | :---: | :---: | :---: |
| $x<x_{1}$ | $x=x_{1}$ | $x>x_{1}$ |  |
| $+$ | $f^{\prime}\left(x_{1}\right)=0$ or is discontinuous | - | Maximum point |
| - | $f^{\prime}\left(x_{1}\right)=0$ or is discontinuous | + | Minimum point |
| $+$ | $f^{\prime}\left(x_{1}\right)=0$ or is discontinuous | $+$ | Neither maximum nor minimum (function increases) |
| - | $f^{\prime}\left(x_{1}\right)=0$ or is discontinuous | - | Neither maximum nor minimum (function decreases) |

Example 1. Test the following function for maximum and minimum:

$$
y=\frac{x^{3}}{3}-2 x^{2}+3 x+1
$$

Solution, 1. Find the first derivative:

$$
y^{\prime}=x^{2}-4 x+3
$$

2. Find the real roots of the derivative:

$$
x^{2}-4 x+3=0
$$

Consequently,

$$
x_{1}=1, \quad x_{2}=3
$$

The derivative is everywhere continuous and so there are no other critical points.
3. Investigate the critical values and record the results in Fig. 107. Investigate the first critical point $x_{1}=1$. Since $y^{\prime}=(x-1)(x-3)$,

$$
\begin{array}{ll}
\text { for } x<1 & \text { we have } y^{\prime}=(-) \cdot(-)>0 \\
\text { for } x>1 & \text { we have } y^{\prime}=(+) \cdot(-)<0
\end{array}
$$

Thus, when passing (from left to right) through the value $x_{1}=1$ the derivative changes sign from plus to minus. Hence, at $x=1$ the function has a maximum, namely,

$$
(y)_{x=1}=\frac{7}{3}
$$

Investigate the second critical point $x_{2}=3$ :

$$
\begin{aligned}
& \text { when } x<3 \text { we have } y^{\prime}=(+) \cdot(-)<0 \\
& \text { when } x>3 \text { we have } y^{\prime}=(+) \cdot(--)>0
\end{aligned}
$$

Thus, when passing through the value $x=3$ the derivative changes sign from minus to plus. Therefore, at $x=3$ the function has a minimum, namely:


Fig. 107

$$
(y)_{x=8}=1
$$

This investigation yiglds the graph of the function (Fig. 107).


Fig. 108

Example 2. Test for maximum and minimum the function

$$
y=(x-1) \sqrt[3]{x^{2}}
$$

Solution. 1. Find the first derivative:

$$
y^{\prime}=\sqrt[3]{x^{2}}+\frac{2(x-1)}{3 \sqrt[3]{x}}=\frac{5 x-2}{3 \sqrt[3]{x}}
$$

2. Find the critical values of the argument: (a) find the points at which the derivative vanishes:

$$
y^{\prime}=\frac{5 x-2}{3 \sqrt[3]{x}}=0, \quad x_{1}=\frac{2}{5}
$$

(b) ind the points at which the derivative becomes discontinuous (in this instance, it becomes infinite). Obviously, that point is

$$
x_{2}=0
$$

(It will be noted that for $x_{2}=0$ the function is defined and continuous.)
There are no other critical points.
3. Investigate the character of the critical points obtained. Investigate the point $x_{1}=\frac{2}{5}$. Noting that

$$
\left(y^{\prime}\right)_{x<\frac{2}{5}}<0, \quad\left(y^{\prime}\right)_{x>\frac{2}{5}}>0
$$

we conclude that at $x=\frac{2}{5}$ the function has a minimum. The value of the function at the minimum point is

$$
(y)_{x=\frac{2}{6}}=\left(\frac{2}{5}-1\right) \sqrt[3]{\frac{4}{25}}=-\frac{3}{5} \sqrt[3]{\frac{4}{25}}
$$

Investigate the second critical point $x=0$. Noting that

$$
\left(y^{\prime}\right)_{x<0}>0, \quad\left(y^{\prime}\right)_{x>0}<0
$$

we conclude that at $x=0$ the function has a maximum, and $(y)_{x=0}=0$. The graph of the investigated function is shown in Fig. 108.

## TESTING A FUNCTION FOR MAXIMUM AND MINIMUM WITH A SECOND DERIVATIVE

Let the derivative of the function $y=f(x)$ vanish at $x=x_{1}$; we have $f^{\prime}\left(x_{1}\right)=0$. Also, let the second derivative $f^{\prime \prime}(x)$ exist and be continuous in some neighbourhood of the point $x_{1}$. Then the following theorem holds.

Theorem. Let $f^{\prime}\left(x_{1}\right)=0$; then at $x=x_{1}$ the function has a maximum if $f^{\prime \prime}\left(x_{1}\right)<0$, and a minimum if $f^{\prime \prime}\left(x_{1}\right)>0$.

Proof. Let us first prove the first part of the theorem. Let

$$
f^{\prime}\left(x_{1}\right)=0 \text { and } f^{\prime \prime}\left(x_{1}\right)<0
$$

Since it is given that $f^{\prime \prime}(x)$ is continuous in some small interval about the point $x=x_{1}$, there will obviously be some small closed interval about the point $x=x_{1}$, at all points of which the second derivative $f^{\prime \prime}(x)$ will be negative.

Since $f^{\prime \prime}(x)$ is the first derivative of the first derivative, $f^{\prime \prime}(x)=$ $\cdots\left(f^{\prime}(x)\right)^{\prime}$, it follows from the condition $\left(f^{\prime}(x)\right)^{\prime}<0$ that $f^{\prime}(x)$ decreases on the closed interval containing $x=x_{1}$ (Sec. 5.2). But $f^{\prime}\left(x_{1}\right)=0$, and so on this interval we have $f^{\prime}(x)>0$ when $x<x_{1}$, und when $x>x_{1}$ we have $f^{\prime}(x)<0$; in other words, the derivalive $f^{\prime}(x)$ changes sign from plus to minus when passing through the point $x=x_{1}$, and this means that at the point $x_{1}$ the function $f(x)$ has a maximum. The first part of the theorem is proved.

The second part of the theorem is proved in similar fashion: If $f^{\prime \prime}\left(x_{1}\right)>0$, then $f^{\prime \prime}(x)>0$ at all points of some closed interval ubout the point $x_{1}$, but then on this interval $f^{\prime \prime}(x)=\left(f^{\prime}(x)\right)^{\prime}>0$ and, hence, $f^{\prime}(x)$ increases. Since $f^{\prime}\left(x_{1}\right)=0$ the derivative $f^{\prime}(x)$ changes sign from minus to plus when passing through the point $x_{1}$, i. e., the function $f(x)$ has a minimum at $x=x_{1}$.

If at the critical point $f^{\prime \prime}\left(x_{1}\right)=0$, then at this point there may be either a maximum or a minimum or neither maximum nor minimum. In this case, investigate by the first method (see Sec. 5.4).

The scheme for investigating extrema with a second derivative is shown in the following table.

| $f^{\prime}\left(x_{1}\right)$ | $f^{\prime \prime}\left(x_{1}\right)$ | Character of critical point |
| :---: | :---: | :---: |
|  |  |  |
| 0 | - | Maximum point <br> 0 |
| 0 | 0 | Minimum point |
| Unknown |  |  |

Example 1. Examine the following function for maximum and minimum

$$
y=2 \sin x+\cos 2 x
$$

Solution. Since the function is periodic with period $2 \pi$, it is sufficient to Investigate the function in the interval $[0,2 \pi]$.

1. Find the derivative:

$$
y^{\prime}=2 \cos x-2 \sin 2 x=2(\cos x-2 \sin x \cos x)=2 \cos x(1-2 \sin x)
$$

2. Find the critical values of the argument:

$$
\begin{gathered}
2 \cos x(1-2 \sin x)=0 \\
x_{1}=\frac{\pi}{6} ; \quad x_{2}=\frac{\pi}{2} ; \quad x_{3}=\frac{5 \pi}{6}, \quad x_{4}=\frac{3 \pi}{2}
\end{gathered}
$$

3. Find the second derivative:

$$
y^{\prime \prime}=-2 \sin x-4 \cos 2 x
$$

4. Investigate the character of each critical point:

$$
\left(y^{\prime \prime}\right)_{x_{1}=\frac{\pi}{6}}=-2 \cdot \frac{1}{2}-4 \cdot \frac{1}{2}=-3<0
$$

Hence, at the point $x_{1}=\frac{\pi}{6}$ we have a maximum:

$$
(y)_{x=\frac{\pi}{8}}=2 \cdot \frac{1}{2}+\frac{1}{2}=\frac{3}{2}
$$

Further,

$$
\left(y^{\prime \prime}\right)_{x=\frac{\pi}{2}}=-2 \cdot 1+4 \cdot 1=2>0
$$

And so at the point $x_{2}=\frac{\pi}{2}$ we have a minimum:

$$
(y)_{x=\frac{\pi}{2}}=2 \cdot 1-1=1
$$

At $x_{3}=\frac{5 \pi}{6}$ we have

$$
\left(y^{\prime \prime}\right)_{x=\frac{5 \pi}{6}}=-2 \cdot \frac{1}{2}-4 \cdot \frac{1}{2}=-3<0
$$

Thus, at $x_{3}=\frac{5 \pi}{6}$ the function has a maximum:

$$
(y)_{x_{3}=\frac{5 \pi}{6}}=2 \cdot \frac{1}{2}+\frac{1}{2}=\frac{3}{2}
$$

Finally,

$$
\left(y^{\prime \prime}\right)_{x=\frac{s \pi}{2}}=-2(-1)-4(-1)=6>0
$$

Consequently, at $x_{4}=\frac{3 \pi}{2}$ we have a minimum:

$$
(y)_{x=\frac{3 \pi}{2}}=2(-1)-1=-3
$$

The graph of the function under investigation is shown in Fig. 109.


Fig. 109
The following examples will show that if at a certain point $x=x_{1}$ we have $f^{\prime}\left(x_{1}\right)=0$ and $f^{\prime \prime}\left(x_{1}\right)=0$, then at this point the function $f(x)$ can have either a maximum or a minimum or neither.

Example 2. Test the following function for maximum and minimum:

$$
y=1-x^{4}
$$

Solution. 1. Find the critical points:

$$
y^{\prime}=-4 x^{3}, \quad-4 x^{3}=0, \quad \dot{x}=0
$$

2. Determine the sign of the second derivative at $x=0$.

$$
y^{\prime \prime}=-12 x^{2}, \quad\left(y^{\prime \prime}\right)_{x=0}==0
$$

It is thus impossible here to determine the character of the critical point liy means of the sign of the second derivative.


Fig. 110


Fig. 111
3. Investigate the character of the critical point by the first method (see Sec. 5.4):

$$
\left(y^{\prime}\right)_{x<0}>0, \quad\left(y^{\prime}\right)_{x>0}<0
$$

Consequently, at $x=0$ the function has a maximum, namely,

$$
(y)_{x=0}=1
$$

The graph of this function is given in Fig. 110.
Example 3. Test for maximum and minimum the function

$$
y=x^{6}
$$

Solution. By the second method we find

$$
\text { 1. } y^{\prime}=6 x^{5}, \quad y^{\prime}=6 x^{5}=0, \quad x=0,
$$

$$
\text { 2. } y^{\prime \prime}=30 x^{4}, \quad\left(y^{\prime \prime}\right)_{x=0}=0
$$

Thus, the second method does not yield anything. Resorting to the first method, we get

$$
\left(y^{\prime}\right)_{x<0}<0, \quad\left(y^{\prime}\right)_{x>0}>0
$$

Therefore, at $x=0$ the function has a minimum (Fig. 111).
Example 4. Test for maximum and minimum the function

$$
y=(x-1)^{3}
$$



Fig. 112

Solution. By the second method we find:

$$
\begin{array}{lc}
y^{\prime}=3(x-1)^{2}, & 3(x-1)^{2}=0, \quad x=1 \\
y^{\prime \prime}=6(x-1), & \left(y^{\prime \prime}\right)_{x=1}=0
\end{array}
$$

Thus, the second method does not yield an answer. By the first method we get

$$
\left(y^{\prime}\right)_{x<1}>0, \quad\left(y^{\prime}\right)_{x>1}>0
$$

Consequently, at $x=1$ the function has neither a maximum nor a minimum (Fig. 112).

## MAXIMUM AND MINIMUM OF. A FUNCTION ON AN INTERVAL

Let a function $y=f(x)$ be continuous on an interval $[a, b]$. Then the function on this interval will have a maximum (see Sec. 2.10). We will assume that on the given interval the function $f(x)$ has a finite number of critical points. If the maximum is reached within the interval $[a, b]$, it is obvious that this value will be one of the maxima of the function (if there are several maxima), namely, the greatest maximum. But it may happen that the maximum value is reached at one of the end points of the interval.

To summarize, then, on the interval $[a, b]$ the function reaches its greatest value either at one of the end points of the interval, or at such an interior point as is the maximum point.

The same may be said about the minimum value of the function: it is attained either at one of the end points of the interval or at an interior point such that the latter is the minimum point.

From the foregoing we get the following rule: if it is required to find the maximum of a continuous function on an interval $[a, b]$, do the following:

1. Find all maxima of the function on the interval.
2. Determine the values of the function at the end points of the interval; that is, evaluate $f(a)$ and $f(b)$.
3. Of all the values of the function obtained choose the greatest; it will be the maximum value of the function on the interval.


Fig. 113

The minimum value of a function on an interval is found in , imilar fashion.

Example. Determine the maximum and minimumof the function $y=x^{3}-3 x+3$ (II) the interval $\left[-3, \frac{3}{2}\right]$.

Solution. 1. Find the maxima and minima of the function on the interval $\left.\mid-3, \frac{3}{2}\right]:$

$$
\begin{aligned}
y^{\prime}=3 x^{2}-3,3 x^{2}-3 & =0, x_{1}=1, x_{2}=-1, \\
y^{\prime \prime} & =6 x
\end{aligned}
$$

then

$$
\left(y^{\prime \prime}\right)_{x=1}=6>0
$$

Hence, there is a minimum at the point $x=1$ :

$$
(y)_{x=1}=1
$$

I urthermore

$$
\left(y^{\prime \prime}\right)_{x x-1}=-6<0
$$

Cionsequently, there is a maximum at the point $x=-1$ :

$$
(y)_{x=-1}=5
$$

2. Determine the value of the function at the end points of the interval:

$$
\text { (y) } x=\frac{3}{2}=\frac{15}{8}, \quad(y)_{x=-3}=-15
$$

Thus, the greatest value of this function on the interval $\left[-3, \frac{3}{2}\right]$ is

$$
(y)_{x=-1}=5
$$

und the smallest value is

$$
(y)_{x=-8}^{-}=-15
$$

The graph of the function is shown in Fig. 113.

## APPLYING THE THEORY OF MAXIMA AND MINIMA OF FUNCTIONS TO THE SOLUTION OF PROBLEMS

The theory of maxima and minima is applied in the solution of many problems of geometry, mechanics, and so forth. Let us examine a few.

Problem 1. The range $R=O A$ (Fig. 114) of a shell (in empty pace) fired with an initial velocity $v_{0}$ from a gun inclined to the horizon at an angle $\varphi$ is determined by the formula

$$
R=\frac{v_{0}^{2} \sin 2 \varphi}{g}
$$

$(k$ is the acceleration of gravity). Determine the angle $\varphi$ at which the range $R$ will be a maximum for a given initial velocity $v_{0}$.

Solution. The quantity $R$ is a function of the variable angle $\varphi$. Test this function for a maximum on the interval $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ :


Fig. 114

$$
\frac{d R}{d \varphi}=\frac{2 v_{0}^{2} \cos 2 \varphi}{g} ; \quad \frac{2 v_{0}^{2} \cos 2 \varphi}{y}=0 ;
$$

critical value $\varphi=\frac{\pi}{4}$;

$$
\frac{d^{2} R}{d \varphi^{2}}=-\frac{4 v_{0}^{2} \sin 2 \varphi}{g}
$$

$$
\left(\frac{d^{2} R}{d \varphi^{2}}\right)_{\varphi=\pi / 4}=-\frac{4 v_{0}^{2}}{g}<0
$$

Hence, for the value $\varphi=\frac{\pi}{4}$ the range $R$ has a maximum:

$$
(R)_{\varphi=\pi / 4}=\frac{v_{0}^{2}}{g}
$$

The values of the function $R$ at the end points of the interval $\left[0, \frac{\pi}{2}\right]$ are

$$
(R)_{\varphi=0}=0, \quad(R)_{\varphi=\pi / 2}=0
$$

Thus, the maximum obtained is the sought-for greatest value of $R$.
Problem 2. What should the dimensions of a cylinder be so that for a given volume $v$ its total surface area $S$ is a minimum?

Solution. Denoting by $r$ the radius of the base of the cylinder and by $h$ the altitude, we have

$$
S=2 \pi r^{2}+2 \pi r h
$$

Since the volume of the cylinder is given, for a given $r$ the quantity $h$ is determined by the formula

$$
v=\pi r^{2} h
$$

whence

$$
h=\frac{r}{\pi r^{2}}
$$

Substituting this expression of $h$ into the formula for $S$, we have

$$
S=2 \pi r^{2}+2 \pi r \frac{v}{\pi r^{2}}
$$

or

$$
S=2\left(\pi r^{2}+\frac{v}{r}\right)
$$

Here $v$ is given, so we have represented $S$ as a function of a single independent variable $r$.

Find the minimum value of this function on the interval () $<r<\infty$ :

$$
\begin{gathered}
\frac{d S}{d r}=2\left(2 \pi r-\frac{v}{r^{2}}\right)^{\bullet} \\
2 \pi r-\frac{v}{r^{2}}=0, \quad r_{1}=\sqrt[3]{\frac{v}{2 \pi}} \\
\left(\frac{d^{2} S}{d r^{2}}\right)_{r=r_{1}}=2\left(2 \pi+\frac{2 v}{r^{3}}\right)_{r=r_{1}}>0
\end{gathered}
$$

Thus, at the point $r=r_{1}$ the function $S$ has a minimum. Noticing that $\lim _{r \rightarrow 0} S=\infty$ and $\lim _{r \rightarrow \infty} S=\infty$, that is, that as $r$ approaches zero or infinity the surface $S$ increases without bound, we arrive at the conclusion that at $r=r_{1}$ the function $S$ has a minimum.

But if $r=\sqrt[3]{\frac{v}{2 \pi}}$, then

$$
h=\frac{v}{\pi r^{2}}=2 \sqrt[3]{\frac{v}{2 \pi}}=2 r
$$

Therefore, for the total surface area $S$ of a cylinder to be a minimum for a given volume $v$, the altitude of the cylinder must be equal to its diameter.

## TESTING A FUNCTION FOR MAXIMUM AND MINIMUM BY MEANS OF TAYLOR'S FORMULA

In Sec. 5.5 , it was noted that if at a certain point $x=a$ we have $f^{\prime}(a)=0$ and $f^{\prime \prime}(a)=0$, then at this point there may be either a maximum or a minimum or neither. And it was noted that in this instance the problem is solved by investigating by the first method; in other words, by testing the sign of the first derivative on the left and on the right of the point $x=a$.

Now we will show that it is possible in this case to investigate by means of Taylor's formula, which was derived in Sec. 4.6.

For greater generality, we assume that not only $f^{\prime \prime}(x)$, but also all derivatives of the function $f(x)$ up to the $n$th order inclusive vanish at $x=a$ :

$$
\begin{equation*}
f^{\prime}(a)=f^{\prime \prime}(a)=\ldots=f^{(n)}(a)=0 \tag{1}
\end{equation*}
$$

and

$$
f^{(n+1)}(a) \neq 0
$$

Further, we assume that $f(x)$ has continuous derivatives up to the $(n+1)$ th order inclusive in the neighbourhood of the point $x=a$.

Write the Taylor formula for $f(x)$, taking account of equalities (1):

$$
\begin{equation*}
f(x)=f(a)+\frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \tag{2}
\end{equation*}
$$

where $\xi$ is a number that lies between $a$ and $x$.
Since $f^{(n+1)}(x)$ is continuous in the neighbourhood of the point $a$ and $f^{(n+1)}(a) \neq 0$, there will be a small positive number $h$ such that for any $x$ that satisfies the inequality $|x-a|<h$, it will be true that $f^{(n+1)}(x) \neq 0$. And if $f^{(n+1)}(a)>0$, then at all points of the interval $\left(a-h, a+h\right.$ ) we will have $f^{(n+1)}(x)>0$; if $f^{(n+1)}(a)<0$, then at all points of this interval we will have $f^{(n+1)}(x)<0$.

Rewrite formula (2) in the form

$$
f(x)-f(a)=\frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)
$$

and consider various special cases.
Case 1. $n$ is odd.
(a) Let $f^{(n+1)}(a)<0$. Then there will be an interval $(a-h, a+h)$ at all points of which the $(n+1)$ th derivative is negative. If $x$ is a point of this interval, then $\xi$ likewise lies between $a-h$ and $a+h$ and, consequently, $f^{(n+1)}(\xi)<0$. Since $n+1$ is an even number, $(x-a)^{n+1}>0$ for $x \neq a$, and therefore the right side of formula $\left(2^{\prime}\right)$ is negative.

Thus, for $x \neq a$ at all points of the interval ( $a-h, a+h$ ) we have

$$
f(x)-f(a)<0
$$

and this means that at $x=a$ the function has a maximum.
(b) Let $f^{(n+1)}(a)>0$. Then we have $f^{(n+1)}(\xi)>0$ for a sufficiently small value of $h$ at all points $x$ of the interval $(a-h, a+h)$. Hence, the right side of formula (2') will be positive; in other words, for $x \neq a$ we will have the following at all points in the given interval:

$$
f(x)-f(a)>0
$$

and this means that at $x=a$ the function has a minimum.
Case 2. $n$ is even.
Then $n+1$ is odd and the quantity $(x-a)^{n+1}$ has different signs for $x<a$ and $x>a$.

If $h$ is sufficiently small in absolute value, then the $(n+1)$ th derivative retains the same sign at all points of the interval $(a-h, a+h)$ as at the point $a$. Thus, $f(x)-f(a)$ has different signs for $x<a$ and $x>a$. But this means that there is neither maximum nor minimum at $x=a$.

It will be noted that if $f^{(n+1)}(a)>0$ when $n$ is even, then $f(x)<f(a)$ for $x<a$ and $f(x)>f(a)$ for $x>a$.

But if $f^{(n+1)}(a)<0$ when $n$ is even, then $f(x)>f(a)$ for $x<a$ and $f(x)<f(a)$ for $x>a$.

The results obtained may be formulated as follows.
If at $x=a$ we have

$$
f^{\prime}(a)=f^{\prime \prime}(a)=\ldots=f^{(n)}(a)=0
$$

und the first nonvanishing derivative $f^{(n+1)}(a)$ is a derivative of even order, then at the point $a$

$$
\begin{aligned}
& f(x) \text { has a maximum if } f^{(n+1)}(a)<0 \\
& f(x) \text { has a minimum if } f^{(n+1)}(a)>0
\end{aligned}
$$

But if the first nonvanishing derivative $f^{(n+1)}(a)$ is a derivative of odd order, then the function has neither maximum nor minimum ut the point $a$. Here,

$$
\begin{aligned}
& f(x) \text { increases if } f^{(n+1)}(a)>0 \\
& f(x) \text { decreases if } f^{(n+1)}(a)<0
\end{aligned}
$$

Example. Test the following function for maximum and minimum:

$$
f(x)=x^{4}-4 x^{3}+6 x^{2}-4 x+1
$$

Solution. We find the critical values of the function

$$
f^{\prime}(x)=4 x^{3}-12 x^{2}+12 x-4=4\left(x^{3}-3 x^{2}+3 x-1\right)
$$

Irom equation

$$
4\left(x^{3}-3 x^{2}+3 x-1\right)=0
$$

we obtain the only critical point

$$
x=1
$$

(since this equation has only one real root).
Investigate the character of the critical point $x=1$ :

$$
\begin{aligned}
f^{\prime \prime}(x)=12 x^{2}-24 x+12=0 & \text { for } x=1 \\
f^{\prime \prime \prime}(x)=24 x-24=0 & \text { for } x=1 \\
f^{\prime V}(x)=24>0 & \text { for any } x
\end{aligned}
$$

Consequently, for $x=1$ the function $f(x)$ has a minimum.

## CONVEXITY AND CONCAVITY OF A CURVE. POINTS OF INFLECTION

In the plane, we consider a curve $y=f(x)$, which is the graph of a single-valued differentiable function $f(x)$.

Definition 1. We say that a curve is convex upreards on the interval $(a, b)$ if all points of the curve lie below any tangent to it on the interval.

We say that the curve is convex downwards on the interval $(b, c)$ if all points of the curve lie above any tangent to it on the interval.

We shall call a curve convex up, a convex curve, and a curve convex down, a concave curve.

Fig. 115 shows a curve convex on the interval $(a, b)$ and concave on the interval $(b, c)$.

An important characteristic of the shape of a curve is its convexity or concavity. This section will be devoted to establishing the characteristics by which, when


Fig. 115 investigating a function $y=f(x)$, one can judge the convexity or concavity (direction of bulge) on various intervals.

We shall prove the following theorem.

Theorem 1. If at all points of an interval $(a, b)$ the second derivative of the function $f(x)$ is negative, i. e., $f^{\prime \prime}(x)<0$, the curve $y=f(x)$ on this interval is convex upreards (the curve is convex).
Proof. In the interval ( $a, b$ ) take an arbitrary point $x=x_{0}$ (Fig. 115) and draw a tangent to the curve at the point with abscissa $x=x_{0}$. The theorem will be proved provided we establish that all the points of the curve on the interval $(a, b)$ lie below this tangent; that is, that the ordinate of any point of the curve $y=f(x)$ is less than the ordinate $y$ of the tangent line for one and the same value of $x$.

The equation of the curve is of the form

$$
\begin{equation*}
y=f(x) \tag{1}
\end{equation*}
$$

But the equation of the tangent to the curve at the point $x=x_{0}$ is of the form

$$
\bar{y}-f\left(x_{0}\right)=f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)
$$

or

$$
\begin{equation*}
\bar{y}=f\left(x_{0}\right)+f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right) \tag{2}
\end{equation*}
$$

From equations (1) and (2) it follows that the difference between the ordinates of the curve and the tangent for the same value of $x$ is

$$
y-\bar{y}=f(x)-f\left(x_{0}\right)-f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)
$$

Applying the Lagrange theorem to the difference $f(x)-f\left(x_{0}\right)$, we get

$$
y-\bar{y}=f^{\prime}(c)\left(x-x_{0}\right)-f^{\prime}\left(x_{0}\right)\left(x-x_{0}\right)
$$

(where $c$ lies between $x_{0}$ and $x$ ) or

$$
y-\bar{y}=\left[f^{\prime}(c)-f^{\prime}\left(x_{0}\right)\right]\left(x-x_{0}\right)
$$

We again apply the Lagrange theorem to the expression in the «quare brackets; then

$$
\begin{equation*}
y-\bar{y}=f^{\prime \prime}\left(c_{1}\right)\left(c-x_{0}\right)\left(x-x_{0}\right) \tag{3}
\end{equation*}
$$

(where $c_{1}$ lies between $x_{0}$ and $c$ ).
Let us first examine the case where $x>x_{0}$. In this case, $x_{0}<$ $\therefore c_{1}<c<x$; since

$$
x-x_{0}>0, \quad c-x_{0}>0
$$

und since, in addition, it is given that

$$
f^{\prime \prime}\left(c_{1}\right)<0
$$

It follows from (3) that $y-\bar{y}<0$.
Now let us consider the case where $x<x_{0}$. In this case $x<c<$ - $c_{1}<x_{0}$ and $x-x_{0}<0, c-x_{0}<0$, and since it is given that $I^{\prime \prime}\left(c_{1}\right)<0$, it follows from (3) that

$$
y-\bar{y}<0
$$

We have thus proved that every point of the curve lies below the tangent to the curve, no matter what values $x$ and $x_{0}$ have on the interval $(a, b)$. And this signifies that the curve is convex. The theorem is proved.
The following theorem is proved in similar fashion.
Theorem 1'. If at all points of the interval (b, c), the second derivative of the function $f(x)$ is positive, that is, $f^{\prime \prime}(x)>0$, then


Fig. 116


Fig. 117

The curve $y=f(x)$ on this interval is convex downwards (the curve is concave).

Note. The content of Theorems 1 and $1^{\prime}$ may be illustrated weometrically. Consider the curve $y=f(x)$, convex upwards on the Interval ( $a, b$ ) (Fig. 116). The derivative $f^{\prime}(x)$ is equal to the linngent of the angle of inclination $\alpha$ of the tangent line at the pinint with abscissa $x$, or $f^{\prime}(x)=\tan \alpha$. For this reason, $f^{\prime \prime}(x)=$ - $(\tan \alpha)_{x}^{\prime}$. It $f^{\prime \prime}(x)<0$ for all $x$ on the interval $(a, b)$, this means
that $\tan \alpha$ decreases with increasing $x$. It is geometrically obvious that if $\tan \alpha$ decreases with increasing $x$, then the corresponding curve is convex. Theorem 1 is an analytic proof of this fact.

Theorem $1^{\prime}$ is illustrated geometrically in similar fashion (Fig. 117).

Example 1. Establish the intervals of convexity and concavity of a curve represented by the equation

$$
y=2-x^{2}
$$

Solution. The second derivative

$$
y^{\prime \prime}=-2<0
$$

for all values of $x$. Hence, the curve is everywhere convex upwards (Fig. 118).
Example 2. The curve is given by the equation
Since

$$
y=e^{x}
$$

$$
y^{\prime \prime}=e^{x}>0
$$

for all values of $x$, the curve is therefore everywhere concave (bulges, or is convex downwards) (Fig. 119).

Example 3. A curve is defined by the equation
Since

$$
y=x^{3}
$$

$$
y^{\prime \prime}=6 x
$$

$y^{\prime \prime}<0$ for $x<0$ and $y^{\prime \prime}>0$ for $x>0$. Hence, for $x<0$ the curve is convex upwards, and for $x>0$, convex down (Fig. 120).


Fig. 118


Fig. 119


Fig. 120

Definition 2. The point that separates the convex part of a continuous curve from the concave part is called the point of inflection of the curve.

In Figs. 120, 121 and 122 the points $O, A$ and $B$ are points of inflection.

It is obvious that at the point of inflection the tangent line, if it exists, cuts the curve, because on one side the curve lies under the tangent and on the other side, above it.

Let us now establish sufficient conditions for a given point of a curve to be a point of inflection.

Theorem 2. Let a curve be defined by an equation $y=f(x)$. If $f^{\prime \prime}(a)=0$ or $f^{\prime \prime}(a)$ does not exist and if the derivative $f^{\prime \prime}(x)$ changes

(a)

(b)

Fig. 121
sign when passing through $x=a$, then the point of the curve with abscissa $x=a$ is the point of inflection.

Proof. (1) Let $f^{\prime \prime}(x)<0$ for $x<a$ and $f^{\prime \prime}(x)>0$ for $x>a$.
Then for $x<a$ the curve is convex up and for $x>a$, it is convex down. Hence, the point $A$ of the curve with abscissa $x=a$ is a point of inflection (Fig. 121).


Fig. 122
(2) If $f^{\prime \prime}(x)>0$ for $x<b$ and $f^{\prime \prime}(x)<0$ for $x>b$, then for $x<b$ the curve is convex down, and for $x>b$, it is convex up. Hence, the point $B$ of the curve with abscissa $x=b$ is a point of inflection (see Fig. 122).

Example 4. Find the points of inflection and determine the intervals of convexity and concavity of the curve

$$
y=e^{-x^{2}} \quad \text { (Gaussian curve) }
$$

Solution. (1) Find the first and second derivatives:

$$
\begin{gathered}
y^{\prime}=-2 x e-x^{2} \\
y^{\prime \prime}=2 e^{-x^{2}}\left(2 x^{2}-1\right)
\end{gathered}
$$

(2) The first and second derivatives exist everywhere. Find the values of $x$ for which $y^{\prime \prime}=0$ :

$$
\begin{gathered}
2 e-x^{2}\left(2 x^{2}-1\right)=0 \\
x_{1}=-\frac{1}{\sqrt{2}}, \quad x_{2}=\frac{1}{\sqrt{2}}
\end{gathered}
$$

(3) Investigate the values obtained:

$$
\begin{aligned}
& \text { for } x<-\frac{1}{\sqrt{2}} \text { we have } y^{\prime \prime}>0 \\
& \text { for } x>-\frac{1}{\sqrt{2}} \text { we have } y^{\prime \prime}<0
\end{aligned}
$$

The second derivative changes sign when passing through the point $x_{1}$. Hence, for $x_{1}=-\frac{1}{\sqrt{2}}$, there is a point of inflection on the curve; its coordinates $\operatorname{are}\left(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$;

$$
\begin{aligned}
& \text { for } x<\frac{1}{\sqrt{2}} \text { we have } y^{\prime \prime}<0 \\
& \text { for } x>\frac{1}{\sqrt{2}} \text { we have } y^{\prime \prime}>0
\end{aligned}
$$

Thus, there is also a point of inflection on the curve for $x_{2}=\frac{1}{\sqrt{2}}$; its coordinates are $\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$. Incidentally, the existence of the second point of inflection follows directly from the symmetry of the curve about the $y$-axis.
(4) From the foregoing it follows that
for $-\infty<x<-\frac{1}{\sqrt{2}}$ the curve is concave:
for $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$ the curve is convex;
for $\frac{1}{\sqrt{2}}<x<+\infty$ the curve is concave
(5) From the expression of the first derivative

$$
y^{\prime}=-2 x e^{-x^{2}}
$$

it follows that
for $x<0 y^{\prime}>0$, the function increases,
for $x>0 y^{\prime}<0$, the function decreases,
for $x=0 \quad y^{\prime}=0$.
At this point the function has a maximum, namely, $y=1$. The foregoing analysis makes it easy to construct a graph of the curve (Fig. 123).

Example 5. Test the curve $y=\boldsymbol{x}^{4}$ for points of inflection.
Solution. (1) Find the second derivative:

$$
y^{\prime \prime}=12 x^{2}
$$

(2) Determine the points at which $y^{\prime \prime}=0$ :

$$
12 x^{2}=0, \quad x=0
$$

(3) Investigate the value $x=0$ obtained:
for $x<0 y^{\prime \prime}>0$, the curve is concave,


Fig 123
for $x>0 y^{\prime \prime}>0$, the curve is concave.
Thus, the curve has no points of inflection (Fig. 124).


Fig. 124


Fig. 125

Example 6. Investigate the following curve for points of inflection:

$$
y=(x-1)^{\frac{1}{3}}
$$

Solution. (1) Find the first and second derivatives:

$$
y^{\prime}=\frac{1}{3}(x-1)^{-\frac{2}{3}} ; y^{\prime \prime}=-\frac{2}{9}(x-1)^{-\frac{5}{3}}
$$

(2) The second derivative does not vanish anywhere, but at $x=1$ it does not exist ( $y^{\prime \prime}= \pm \infty$ ).
(3) Investigate the value $x=1$ :

$$
\begin{array}{lll}
\text { for } x<1 & y^{\prime \prime}>0, & \text { the curve is concave; } \\
\text { for } x>1 & y^{\prime \prime}<0, & \text { the curve is convex. }
\end{array}
$$

Consequently, at $x=1$ there is a point of inflection $(1,0)$.
It will be noted that for $x=1 y^{\prime}=\infty$; the curve at this point has a vertical tangent (Fig. 125).

### 5.10 ASYMPTOTES

Very frequently one has to investigate the shape of a curve $y=f(x)$ and, consequently, the type of variation of the corresponding function in the case of an unlimited increase (in absolute value) of the abscissa or ordinate of a variable point of the curve, or of the abscissa and ordinate simultaneously. Here, an important special case is when the curve under study approaches a given line without bound as the variable point of the curve recedes to infinity.*


Fig. 126


Fig. 127

Definition. A straight line $A$ is called an asymptote to a curve, if the distance $\delta$ from the variable point $M$ of the curve to this straight line approaches zero as the point $M$ recedes to infinity (Figs. 126 and 127).

In future we shall differentiate between vertical asymptotes (parallel to the axis of ordinates) and inclined asymptotes (not parallel to the axis of ordinates).

1. Vertical asymptotes. From the definition of an asymptote it follows that if $\lim _{x \rightarrow a+0} f(x)=\infty$ or $\lim _{x \rightarrow a-0} f(x)=\infty$ or $\lim _{x \rightarrow a} f(x)=\infty$, then the straight line $x=a$ is an asymptote to the curve $y=f(x)$; and, conversely, if the straight line $x=a$ is an asymptote, then one of the foregoing equalities is fulfilled.

Consequently, to find vertical asymptotes one has to find values of $x=a$ such that when they are approached by the function $y=f(x)$ the latter approaches infinity. Then the straight line $x=a$ will be a vertical asymptote.

Example 1. The curve $y=\frac{2}{x-5}$ has a vertical asymptote $x=5$, since $y \rightarrow \infty$ as $x \longrightarrow 5$ (Fig. 128).

[^1]Example 2. The curve $y=\tan x$ has an Infinite number of vertical asymptotes

$$
x= \pm \frac{\pi}{2}, \quad x= \pm \frac{3 \pi}{2}, \quad x= \pm \frac{5 \pi}{2}, \ldots
$$

This follows from the fact that $\tan x \rightarrow \infty$ нS $x$ approaches the values $\frac{\pi}{2}, \frac{3 \pi}{2}, \frac{5 \pi}{2}, \ldots$, or $-\frac{\pi}{2},-\frac{3 \pi}{2},-\frac{5 \pi}{2}, \ldots$ (Fig. 129).

Example 3. The curve $y=e^{\frac{1}{x}}$ has a vertiral asymptote $x=0$, since $\lim _{x \rightarrow+0} e^{\frac{1}{x}}=\infty$


Fig. 128 (lig. 130).
2. Inclined asymptotes. Let the curve $y=f(x)$ have an inclined usymptote whose equation is

$$
\begin{equation*}
y=k x+b \tag{1}
\end{equation*}
$$

Determine the numbers $k$ and $b$ (Fig. 131). Let $M(x, y)$ be a point lying on the curve and $N(x, \bar{y})$, a point lying on the asymptote.


Fig. 129
The length of $M P$ is equal to the distance from the point $M$ to the asymptote. It is given that

$$
\begin{equation*}
\lim _{x \rightarrow+\infty} M P=0 \tag{2}
\end{equation*}
$$

Designating the angle of inclination of the asymptote to the $x$-axis by $\varphi$, we find from $\triangle N M P$ that

$$
N M=\frac{M P}{\cos \varphi}
$$

Since $\varphi$ is a constant angle (not equal to $\frac{\pi}{2}$ ), by virtue of the foregoing equation

$$
\lim _{x \rightarrow+\infty} N M=0
$$

and, conversely, from (2') we get (2). But

$$
N M=|Q M-Q N|=|y-\bar{y}|=|f(x)-(k x+b)|
$$

and ( $2^{\prime}$ ) takes the form

$$
\begin{equation*}
\lim _{x \rightarrow+\infty}[f(x)-k x-b]=0 \tag{3}
\end{equation*}
$$

To summarize: if the straight line (1) is an asymptote, then (3) is satisfied, and conversely, if, $k$ and $b$ are constant, equation (3) is


Fig. 130


Fig. 131
satisfied, then the straight line $y=k x+b$ is an asymptote. Let us now define $k$ and $b$. Taking $x$ outside the brackets in (3), we get

$$
\lim _{x \rightarrow+\infty} x\left[\frac{f(x)}{x}-k-\frac{b}{x}\right]=0
$$

Since $x \rightarrow+\infty$, the following equation must hold true:

$$
\lim _{x \rightarrow+\infty}\left[\frac{f(x)}{x}-k-\frac{b}{x}\right]=0
$$

For $b$ constant, $\lim _{x \rightarrow \infty} \frac{b}{x}=0$. Hence,

$$
\lim _{x \rightarrow+\infty}\left[\frac{f(x)}{x}-k\right]=0
$$

or

$$
\begin{equation*}
k=\lim _{x \rightarrow+\infty} \frac{f(x)}{x} \tag{4}
\end{equation*}
$$

Knowing $k$, we find $b$ from (3):

$$
\begin{equation*}
b=\lim _{x \rightarrow+\infty}[f(x)-k x] \tag{5}
\end{equation*}
$$

Thus, if the straight line $y=k x+b$ is an asymptote, then $k$ and $b$ may be found from (4) and (5). Conversely, if the limits (4) and (5) exist, then (3) is fulfilled and the straight line $y=k x+b$ is an asymptote. If even one of the limits (4) or (5) does not exist, then the curve does not have an asymptote.

It should be noted that we carried out our investigation as applied to Fig. 131, as $x \rightarrow+\infty$, but all the arguments hold also for the case $x \rightarrow-\infty$.

Example 4. Find the asymptotes of the curve

$$
y=\frac{x^{2}+2 x-1}{x}
$$

Solution. (1) Look for vertical asymptotes:
when $x \rightarrow-0 \quad y \longrightarrow+\infty$
when $x \rightarrow+0 \quad y \rightarrow-\infty$
Therefore, the straight line $x=0$ is a vertical asymptote.
(2) Look for inclined asymptotes:

$$
\begin{aligned}
& k=\lim _{x \rightarrow \pm \infty} \frac{y}{x}=\lim _{x \rightarrow \pm \infty} \frac{x^{2}+2 x-1}{x^{2}}= \\
& \therefore \lim _{x \rightarrow \pm \infty}\left[1+\frac{2}{x}-\frac{1}{x^{2}}\right]=1
\end{aligned}
$$



Fig. 132
that is,

$$
\begin{gathered}
b=\lim _{x \rightarrow \pm \infty}[y-x]=\lim _{x \rightarrow \pm \infty}\left[\frac{x^{2}+2 x-1}{x}-x\right]=\lim _{x \rightarrow \pm x}\left[\frac{x^{2}+2 x-1-x^{2}}{x}\right] \\
=\lim _{x \rightarrow \pm \infty}\left[2-\frac{1}{x}\right]=2
\end{gathered}
$$

or, finally,

$$
b=2
$$

Therefore, the straight line

$$
y=x+2
$$

Is an inclined asymptote to the given curve.
To investigate the mutual positions of a curve and an asymptote, let us consider the difference of the ordinates of the curve and the asymptote for
one and the same value of $x$ :

$$
\frac{x^{2}+2 x-1}{x}-(x+2)=-\frac{1}{x}
$$

This difference is negative for $x>0$ and positive for $x<0$; and so for $x>0$ the curve lies below the asymptote, and for $x<0$ it lies above the asymptote (Fig. 132).

Example 5. Find the asymptotes of the curve

$$
y=e^{-x} \sin x+x
$$

Solution. (1) It is obvious that there are no vertical asymptotes.
(2) Look for inclined asymptotes:

$$
\begin{gathered}
k=\lim _{x \rightarrow+\infty} \frac{y}{x}=\lim _{x \rightarrow+\infty} \frac{e^{-x} \sin x+x}{x}=\lim _{x \rightarrow+\infty}\left[\frac{e^{-x} \sin x}{x}+1\right]=1 \\
b=\lim _{x \rightarrow+\infty}\left[e^{-x} \sin x+x-x\right]=\lim _{x \rightarrow+\infty} e^{-x} \sin x=0
\end{gathered}
$$

Hence, the straight line $y=x$ is an inclined asymptote as $x \longrightarrow+\infty$.
The given curve has no asymptote as $x \rightarrow-\infty$. Indeed, the limit $\lim _{x \rightarrow-\infty} \frac{y}{x}$ does not exist, since $\frac{y}{x}=\frac{e^{-x}}{x} \sin x+1$. (Here, the first term increases without bound as $x \longrightarrow-\infty$ and, therefore it has no limit.)

## general plan for investigating functions and constructing graphs

The term "investigation of a function" usually implies the finding of:
(1) the natural domain of the function;
(2) the discontinuities of the function;
(3) the intervals of increase and decrease of the function;
(4) the maximum point and the minimum point, and also the maximal and minimal values of the functions;
(5) the regions of convexity and concavity of the graph, and points of inflection;
(6) the asymptotes of the graph of the function.

The graph of the function is constructed on the basis of such an investigation (it is sometimes wise to plot certain elements of the graph in the very process of investigation).

Note 1. If the function under investigation $y=f(x)$ is even, that is, such that upon a change in sign of the argument the value of the function does not change, i.e., if

$$
f(-x)=f(x)
$$

then it is sufficient to investigate the function and construct its graph for positive values of the argument that lie within the domain of definition of the function. For negative values of the argument, the graph of the function is constructed on the grounds
that the graph of an even function is symmetric about the ordinate axis.

Example 1. The function $y=x^{2}$ is even, since ( $\left.-x\right)^{2}=x^{2}$ (see Fig. 5).
Example 2. The function $y=\cos x$ is even, since $\cos (-x)=\cos x$ (see lig. 16).

Note 2. If the function $y=f(x)$ is odd, that is, such that for uny change in the argument the function changes sign, i.e., if

$$
f(-x)=-f(x)
$$

then it is sufficient to investigate this function in the case of positive values of the argument. The graph of an odd function is symmetric about the origin.

Example 3. The function $y=x^{3}$ is odd, since $(-x)^{3}=-x^{3}$ (see Fig. 7).
Example 4. The function $y=\sin x$ is odd, since $\sin (-x)=-\sin x$ (see lig. 15).

Note 3. Since a knowledge of certain properties of a function nllows us to judge of the other properties, it is sometimes advisable to choose the order of investigation on the basis of the peculiarities of the given function. For example, if we have found out that the given function is continuous and differentiable and if we have found the maximum point and the minimum point of this function, we have thus already determined also the range of increase and decrease of the function.

Example 5. Investigate the function

$$
y=\frac{x}{1+x^{2}}
$$

nind construct its graph.
Solution. (1) The domain of the function is the interval $-\infty<x<+\infty$. Il will straightaway be noted that for $x<0$ we have $y<0$, and for $x>0$ we liave $y>0$.
(2) The function is everywhere continuous.
(3) Test the function for maximum and minimum: from the equation

$$
y^{\prime}=\frac{1-x^{2}}{\left(1+x^{2}\right)^{2}}=0
$$

nnd the critical points:

$$
x_{1}=-1, \quad x_{2}=1
$$

Investigate the character of the critical points:

$$
\begin{aligned}
& \text { for } x<-1 \text { we have } y^{\prime}<0 \\
& \text { for } x>-1 \text { we have } y^{\prime}>0
\end{aligned}
$$

Hence, at $x=-1$ the function has a minimum:

$$
y_{\min }=(y)_{x=-1}=-0.5
$$

I urthermore
for $x<1$ we have $y>0$
for $x>1$ we have $y^{\prime}<0$

Hence, at $x=1$ the function has a maximum:

$$
y_{\max }=(y)_{x=1}=0.5
$$

(4) Determine the domains of increase and decrease of the function:
for $-\infty<x<-1$ we have $y^{\prime}<0$, the function decreases, for $-1<x<1$ we have $y^{\prime}>0$, the function increases, for $1<x<+\infty$ we have $y^{\prime}<0$, the function decreases.
(5) Determine the domains of convexity and concavity of the curve and the points of inflection: from the equation

$$
y^{n}=\frac{2 x\left(x^{2}-3\right)}{\left(1+x^{2}\right)^{3}}=0
$$

we get

$$
x_{1}=-\sqrt{3}, x_{2}=0, x_{3}=\sqrt{3}
$$

Investigating $y^{\prime \prime}$ as a function of $x$ we find that

$$
\text { for }-\infty<x<-\sqrt{3} y^{\prime \prime}<0 \text {, the curve is convex, }
$$

$$
\text { for }-\sqrt{3}<x<0 \quad y^{\prime \prime}>0 \text {, the curve is concave, }
$$

$$
\text { for } 0<x<\sqrt{3} y^{\prime \prime}<0, \text { the curve is convex, }
$$

$$
\text { for } \sqrt{3}<x<+\infty \quad y^{\prime \prime}>0 \text {, the curve is concave. }
$$

Thus, the point with coordinates $x=-\sqrt{3}, y=-\frac{\sqrt{3}}{4}$ is a point of inflection; in exactly the same way, the points $(0,0)$ and $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$ are points of inflection.
(6) Determine the asymptotes of the curve:

$$
\begin{array}{ll}
\text { for } x \rightarrow+\infty & y \longrightarrow 0 \\
\text { for } x \rightarrow-\infty & y \longrightarrow 0
\end{array}
$$

Consequently, the straight line $y=0$ is the only inclined asymptote. The curve has no vertical asymptotes because the function does not approach infinity for a single finite value of $x$.


Fig. 133

The graph of the curve under study is given in Fig. 133.
Example 6. Investigate the function

$$
y=\sqrt[3]{2 a x^{2}-x^{3}} \quad(a>0)
$$

and construct its graph.
Solution. (1) The function is defined for all values of $x$.
(2) The function is everywhere continuous.
(3) Test the function for maximum and minimum:

$$
y^{\prime}=\frac{4 a x-3 x^{2}}{3 \sqrt[3]{\left(2 a x^{2}-x^{2}\right)^{2}}}=\frac{4 a-3 x}{3 \sqrt[3]{x(2 a-x)^{2}}}
$$

There is a derivative everywhere except at the points

$$
x_{1}=0 \text { and } x_{2}=2 a
$$

Investigate the limiting values of the derivative as $x \longrightarrow-0$ and as $x \longrightarrow+0$ :

$$
\lim _{x \rightarrow-0} \frac{4 a-3 x}{3 \sqrt[3]{x} \sqrt[3]{(2 a-x)^{2}}}=-\infty, \quad \lim _{x \rightarrow+0} \frac{4 a-3 x}{3 \sqrt[3]{x} \sqrt[3]{(2 a-x)^{2}}}=+\infty
$$

for $x<0 y^{\prime}<0$, and for $x>0 y^{\prime}>0$.
Hence, at $x=0$ the function has a minimum. The value of the function at this point is zero.

Now investigate the function at the other critical point $x_{2}=2 a$ As $x \longrightarrow 2 a$ the derivative also approaches infinity. However, in this case, for all values of $x$ close to $2 a$ (both on the right and left of $2 a$ ), the derivative is negative. Therefore, at this point the function has neither a maximum nor a minimum. $\Lambda t$ and about the point $x_{z}=2 a$ the function decreases; the tangent to the curve at this point is vertical.

At $x=\frac{4 a}{3}$ the derivative vanishes. Let us investigate the character of this critical point. Examining the expression of the first derivative, we note that

$$
\text { for } x<\frac{4 a}{3} \quad y^{\prime}>0, \text { and for } x>\frac{4 a}{3} \quad y^{\prime}<0
$$

Thus, at $x=\frac{4 a}{3}$ the function has a maximum:

$$
y_{\max }=\frac{2}{3} a \sqrt[3]{4}
$$

(4) On the basis of this study we get the domains of increase and decrease of the function:
for $-\infty<x<0$ the function decreases,
for $0<x<\frac{4 a}{3}$ the function increases,
for $\frac{4 a}{3}<x<+\infty$ the function decreases.
(5) Determine the domains of convexity and concavity of the curve and the points of inflection: the second derivative

$$
y^{\prime \prime}=-\frac{8 a^{2}}{9 x^{\frac{4}{3}}(2 a-x)^{\frac{6}{3}}}
$$

does not vanish at a single point. Yet there are two points at which the second derivative is discontinuous: $x_{1}=0$ and $x_{2}=2 a$.

Let us investigate the sign of the second derivative near each of these points. For $x<0$ we have $y^{\prime \prime}<0$ and the curve is convex up; for $x>0$ we have $y^{\prime \prime}<0$ and the curve is convex up. Hence, the point with abscissa $x=0$ is not a point of inflection.

For $x<2 a$ we have $y^{\prime \prime}<0$ and the curve is convex up; for $x>2 a$ we have $y^{\prime \prime}>0$ and the curve is convex down. Hence, the point $(2 a, 0)$ on the curve is a point of inflection.
(6) Determine the asymptotes of the curve:

$$
\begin{gathered}
k=\lim _{x \rightarrow \pm \infty} \frac{y}{x}=\lim _{x \rightarrow \pm \infty} \frac{\sqrt[3]{2 a x^{2}-x^{3}}}{x}=\lim _{x \rightarrow \pm \infty} \sqrt[3]{\frac{2 a}{x}-1}=-1 \\
b=\lim _{x \rightarrow \pm \infty}\left[\sqrt[3]{2 a x^{2}-x^{3}}+x\right]=\lim _{x \rightarrow \pm \infty} \frac{2 a x^{2}-x^{3}+x^{3}}{\sqrt[3]{\left(2 a x^{2}-x^{3}\right)^{2}}-x \sqrt[3]{2 a x^{2}-x^{3}}+x^{2}}=\frac{2 a}{3}
\end{gathered}
$$

Thus the straight line

$$
y=-x+\frac{2 a}{3}
$$

is an inclined asymptote to the curve $y=\sqrt[3]{2 a x^{2}-x^{3}}$. The graph of this func-


Fig. 134

INVESTIGATING CURVES REPRESENTED PARAMETRICALLY

Let a curve be given by the parametric equations

$$
\left.\begin{array}{l}
x=\varphi(t)  \tag{1}\\
y=\psi(t)
\end{array}\right\}
$$

In this case the investigation and construction of the curve is carried out just as for the curve given by the equation

$$
y=f(x)
$$

Evaluate the derivatives

$$
\left.\begin{array}{l}
\frac{d x}{d t}=\varphi^{\prime}(t)  \tag{2}\\
\frac{d y}{d t}=\psi^{\prime}(t)
\end{array}\right\}
$$

For those points of the curve near which it is the graph of a certain function $y=f(x)$, evaluate the derivative

$$
\begin{equation*}
\frac{d y}{d x}=\frac{\psi^{\prime}(t)}{\varphi^{\prime}(t)} \tag{3}
\end{equation*}
$$

We find the values of the parameter $t=t_{1}, t_{2}, \ldots, t_{k}$ for which at least one of the derivatives $\varphi^{\prime}(t)$ or $\psi^{\prime}(t)$ vanishes or becomes discontinuous. (We shall call these values of $t$ critical values.)

By formula (3), in each of the intervals $\left(t_{1}, t_{2}\right) ;\left(t_{2}, t_{3}\right) ; \ldots$; $\left(t_{k-1}, t_{k}\right)$ and hence, in each of the intervals $\left(x_{1}, x_{2}\right) ;\left(x_{2}, x_{3}\right)$; $\ldots ;\left(x_{k-1}, x_{k}\right)$ [where $x_{i}=\varphi\left(t_{i}\right)$ ], we determine the sign of $\frac{d y}{d x}$, in this way determining the domain of increase and decrease. This likewise enables us to determine the character of points that correspond to the values of the parameter $t_{1}, t_{2}, \ldots, t_{k}$. Next, we compute

$$
\begin{equation*}
\frac{d^{2} y}{d x^{2}}=\frac{\psi^{\prime \prime}(t) \varphi^{\prime}(t)-\varphi^{\prime \prime}(t) \psi^{\prime}(t)}{\left[\varphi^{\prime}(t)\right]^{3}} \tag{4}
\end{equation*}
$$

From this formula, we determine the direction of convexity of the curve at each point.

To find the asymptotes determine those values of $t$, upon approach to which either $x$ or $y$ approaches infinity, and those values of $t$ upon approach to which both $x$ and $y$ approach infinity. Then carry out the investigation in the usual way.

The following examples will serve to illustrate some of the peculiarities that appear when investigating curves represented parametrically.

Example 1. Investigate the curve given by the equation

$$
\left.\begin{array}{l}
x=a \cos ^{3} t \\
y=a \sin ^{3} t
\end{array}\right\} \quad(a>0)
$$

Solution. The quantities $x$ and $y$ are defined for all values of $t$. But since he functions $\cos ^{3} t$ and $\sin ^{3} t$ are periodic, of a period $2 \pi$, it is sufficient to consider the variation of the parameter $t$ in the range from 0 to $2 \pi$; here the nterval $[-a, a]$ is the range of $x$ and the interval $[-a ; a]$ is the range of $y$. Consequently, this curve has no asymptotes. Next, we find

$$
\left.\begin{array}{l}
\frac{d x}{d t}=-3 a \cos ^{2} t \sin t \\
\frac{d y}{d t}=3 a \sin ^{2} t \cos t
\end{array}\right\}
$$

These derivatives vanish at $t=0, \frac{\pi}{2}, \pi, \frac{3 \pi}{2}, 2 \pi$. We determine

$$
\frac{d y}{d x}=\frac{3 a \sin ^{2} t \cos t}{-3 a \cos ^{2} t \sin t}=-\tan t
$$

On the basis of $\left(2^{\prime}\right)$ and $\left(3^{\prime}\right)$ we compile the following table:

| Range of $t$ | Corresponding <br> range of $x$ | Corresponding <br> range of $y$ | Sign <br> of $\frac{d y}{d x}$ | Type of varia- <br> tion of $y$ as a a <br> functon of <br> $x[y=f(x)]$ |
| :--- | :--- | :--- | :--- | :--- |
| $0<t<\frac{\pi}{2}$ | $a>x>0$ | $0<y<a$ | - | Decreases |
| $\frac{\pi}{2}<t<\pi$ | $0>x>-a$ | $a>y>0$ | + | Increases |
| $\pi<t<\frac{3 \pi}{2}$ | $-a<x<0$ | $0>y>-a$ | - | Decreases |
| $\frac{3 \pi}{2}<t<2 \pi$ | $0<x<a$ | $-a<y<0$ | + | Increases |

From the table it follows that equations ( $1^{\prime}$ ) denne two continuous functions of the type $y=f(x)$, for $0 \leqslant t \leqslant \pi y \geqslant 0$ (see first two lines of the table), for $\pi<t \leqslant 2 \pi y<0$ (see last two lines of the table). From ( $3^{\prime}$ ) it follows that

$$
\lim _{t \rightarrow \frac{\pi}{2}} \frac{d y}{d x}=\infty
$$

and

$$
\lim _{x \rightarrow \frac{3 \pi}{2}} \frac{d y}{d x}=\infty
$$

At these points the tangent to the curve is vertical. We now find

$$
\left.\frac{d y}{d t}\right|_{t=0}=0,\left.\quad \frac{d y}{d t}\right|_{t=\pi}=0,\left.\frac{d y}{d t}\right|_{t=2 \pi}=0
$$

At these points the tangent to the curve is horizontal. We then find


Fig. 135

$$
\frac{d^{2} y}{d x^{2}}=\frac{1}{3 a \cos ^{4} t \sin t}
$$

Whence it follows that
for $0<t<\pi, \frac{d^{2} y}{d x^{2}}>0$, the curve is concave,
for $\pi<t<2 \pi, \frac{d^{2} y}{d x^{2}}<0$, the curve is convex.
On the basis of this investigation we can construct a curve (Fig. 135), which is called an astroid.

Example 2. Construct a curve given by the following equations (folium of Descartes):

$$
x=\frac{3 a t}{1+t^{3}}, \quad y=\frac{3 a t^{2}}{1+t^{3}} \quad(a>0)
$$

Solution. Both functions are defined for all values of $t$ except at $t=-1$, and

$$
\begin{aligned}
& \lim _{t \rightarrow-1-0} x=\lim _{t \rightarrow-1-0} \frac{3 a t}{1+t^{3}}=+\infty, \lim _{t \rightarrow-1+0} x=-\infty, \\
& \lim _{t \rightarrow-1-0} y=\lim _{t \rightarrow-1-0} \frac{3 a t^{2}}{1+t^{3}}=-\infty, \lim _{t \rightarrow-1+0} y=+\infty,
\end{aligned}
$$

Further note that

$$
\begin{array}{lll}
\text { when } t=0 & x=0, & y=0 \\
\text { when } t \longrightarrow+\infty & x \longrightarrow 0, & y \longrightarrow 0 \\
\text { when } t \longrightarrow-\infty & x \longrightarrow 0, & y \longrightarrow 0
\end{array}
$$

Find $\frac{d x}{d t}$ and $\frac{d y}{d t}$ :

$$
\frac{d x}{d t}=\frac{6 a\left(\frac{1}{2}-t^{3}\right)}{\left(1+t^{3}\right)^{2}}, \quad \frac{d y}{d t}=\frac{3 a t\left(2-t^{3}\right)}{\left(1+t^{3}\right)^{2}}
$$

For the parameter $t$ we get the following four critical values:

$$
t_{1}=-1, \quad t_{2}=0, \quad t_{3}=\frac{1}{\sqrt[3]{2}}, \quad t_{4}=\sqrt[3]{2}
$$

Then we find

$$
\frac{d y}{d x}=\frac{\frac{d y}{d t}}{\frac{d x}{d t}}=\frac{t\left(2-t^{3}\right)}{2\left(\frac{1}{2}-t^{3}\right)}
$$

On the basis of formulas ( $1^{\prime \prime}$ ), $\left(2^{\prime \prime}\right)$, and $\left(3^{\prime \prime}\right)$ we compile the following table:

| Range of $t$ | Corresponding range of $\boldsymbol{x}$ | Corresponding range of $y$ | $\begin{aligned} & \text { Sign } \\ & \text { of } \frac{d y}{d x} \end{aligned}$ | Type of varia thon of $y$ as a function of $x[y=f(x)]$ |
| :---: | :---: | :---: | :---: | :---: |
| $-\infty<t<-1$ | $0<x<+\infty$ | $0>y>-\infty$ | - | Decreases |
| $-1<t<0$ | $-\infty<x<0$ | $+\infty>y>0$ | - | Decreases |
| $0<t<\frac{1}{\sqrt[3]{2}}$ | $0<x<a \sqrt[3]{4}$ | $0<y<a \sqrt[3]{2}$ | + | Increases |
| $\frac{1}{\sqrt[3]{2}}<t<\sqrt[3]{2}$ | $a \sqrt[3]{4}>x>a \sqrt[3]{2}$ | $a \sqrt[3]{2}<y<a \sqrt[3]{4}$ | - | Decreases |
| $\sqrt[3]{2}<t<+\infty$ | $a \sqrt[3]{2}>x>0$ | $a \sqrt[3]{4}>y>0$ | + | Increases |

From (3") we find

$$
\left(\frac{d y}{d x}\right)_{\substack{t=0 \\(x=0 \\ y=0}}=0, \quad\left(\frac{d y}{d x}\right)_{\substack{t=\infty \\(x=0 \\ y=0}}=\infty
$$

Thus, the curve cuts the origin twice: with the tangent parallel to the $x$-axis and with the tangent parallel to the $y$-axis. Further,


Fig. 136

$$
\begin{gathered}
\left(\frac{d y}{d x}\right)_{t=\frac{1}{\sqrt[3]{2}}}=\infty \\
\binom{x=a \sqrt[3]{4}}{y=a \sqrt[3]{2}}
\end{gathered}
$$

At this point the tangent to the curve is vertical.

$$
\begin{gathered}
\left(\frac{d y}{d x}\right)_{t=\sqrt[3]{2}}=0 \\
\binom{\mid x=a \sqrt[3]{2}}{y=a \sqrt[3]{4}}
\end{gathered}
$$

At this point the tangent to the curve is horizontal. Let us investigate the question of the existence of an asymptote:

$$
\begin{gathered}
k=\lim _{x \rightarrow+\infty} \frac{y}{x}=\lim _{t \rightarrow-1-0} \frac{3 a t^{2}\left(1+t^{3}\right)}{3 a t\left(1+t^{3}\right)}=-1 \\
b=\lim _{x \rightarrow+\infty}(y-k x)=\lim _{x \rightarrow-1-0}\left[\frac{3 a t^{2}}{1+t^{3}}-(-1) \frac{3 a t}{1+t^{3}}\right]= \\
=\lim _{t \rightarrow-1-0}\left[\frac{3 a t(t+1)}{1+t^{3}}\right]=\lim _{t \rightarrow-1-0} \frac{3 a t}{1-t+t^{2}}=-a
\end{gathered}
$$

Hence, the straight line $y=-x-a$ is an asymptote to a branch of the curve as $x \rightarrow+\infty$.

Similarly we find

$$
k=\lim _{x \rightarrow-\infty} \frac{y}{x}=-1, \quad b=\lim _{x \rightarrow-\infty}(y-k x)=-a
$$

Thus, the straight line is also an asymptote to a branch of the curve as $x \rightarrow-\infty$.

On the basis of this investigation we construct the curve (Fig. 136).
Some problems involving investigation of curves will again be discussed in Sec. 8.20 ("Singular Points of a Curve").


[^0]:    * This definition is sometimes formulated as follows: a function $f(x)$ has n maximum at $x_{1}$ if it is possible to find a neighbourhood $(\alpha, \beta)$ of $x_{1}\left(\alpha<x_{1}<\beta\right)$ anch that for all points of this neighbourhood different from $x_{1}$ the inequality $f(x)<f\left(x_{1}\right)$ is fulfilled.

[^1]:    * We say the variable point $M$ moves along a curve to infinity if the distance of the point from the origin increases without bound.

