# SOME THEOREMS ON DIFFERENTIABLE FUNCTIONS 

## A THEOREM ON THE ROOTS OF A DERIVATIVE (ROLLE'S THEOREM)

Rolle's Theorem. If a function $f(x)$ is continuous on an interval [ $a, b]$ and is differentiable at all interior points of the interval, and vanishes $[f(a)=f(b)=0$ ] at the end points $x=a$ and $x=b$, then inside $[a, b]$ there exists at least one point $x=c, a<c<b$, "t which the derivative $f^{\prime}(x)$ vanishes, that is, $f^{\prime}(c)=0$.*

Proof. Since the function $f(x)$ is continuous on the interval $[a, b]$, it has a maximum $M$ and a minimum $m$ on that interval.

If $M=m$ the function $f(x)$ is constant, which means that for "II values of $x$ it has a constant value $f(x)=m$. But then at any point of the interval $f^{\prime}(x)=0$, and the theorem is proved.

Suppose $M \neq m$. Then at least one of these numbers is not igual to zero.

For the sake of definiteness, let us assume that $M>0$ and that The function takes on its maximum value at $x=c$, so that $f(c)=M$. Let it be noted that, here, $c$ is not equal either to $a$ or to $b$, since it is given that $f(a)=0, f(b)=0$. Since $f(c)$ is the maximum value of the function, it follows that $f(c+\Delta x)-$
$f(c) \leqslant 0$, both when $\Delta x>0$ and when $\Delta x<0$. Whence it follows that

$$
\begin{align*}
& \frac{f(c+\Delta x)-f(c)}{\Delta x} \leqslant 0 \text { when } \Delta x>0 \\
& \frac{f(c+\Delta x)-f(c)}{\Delta x} \geqslant 0 \text { when } \Delta x<0
\end{align*}
$$

Since it is given in the theorem that the derivative at $x=c$ rxists, we get, upon passing to the limit as $\Delta x \rightarrow 0$,

$$
\begin{array}{lll}
\lim _{\Delta x \rightarrow 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}=f^{\prime}(c) \leqslant 0 & \text { when } & \Delta x>0 \\
\lim _{\Delta x \rightarrow 0} \frac{f(c+\Delta x)-f(c)}{\Delta x}=f^{\prime}(c) \geqslant 0 & \text { when } & \Delta x<0
\end{array}
$$

But the relations $f^{\prime}(c) \leqslant 0$ and $f^{\prime}(c) \geqslant 0$ are compatible only if $f^{\prime}(c)=0$. Consequently, there is a point $c$ inside the interval $[a, b]$ at which the derivative $f^{\prime}(x)$ is equal to zero.

[^0]The theorem about the roots of a derivative has a simple geometric interpretation: if a continuous curve, which at each point has a tangent, intersects the $x$-axis at points with abscissas $a$ and $b$, then on this curve there will be at least one point with abscissa $c, a<c<b$, at which the tangent is parallel to the $x$-axis.


Fig. 92


Fig. 93

Note 1. The theorem that has just been proved also holds for a differentiable function such that does not vanish at the end points of the interval $[a, b]$, but takes on equal values $f(a)=f(b)$ (Fig. 92). The proof in this case is exactly the same as before.

Note 2. If the function $f(x)$ is such that the derivative does not exist at all points within the interval $[a, b]$, the assertion of the theorem may prove erroneous (in this case there might not be a point $c$ in the interval $[a, b]$, at which the derivative $f^{\prime}(x)$ vanishes).

For example, the function

$$
y=f(x)=1-\sqrt[3]{x^{2}}
$$

(Fig. 93) is continuous on the interval $[-1,1]$ and vanishes at the end points of the interval, yet the derivative

$$
f^{\prime}(x)=-\frac{2}{3 \sqrt[3]{x}}
$$

within the interval does not vanish. This is because there is a point $x=0$ inside the interval at which the derivative does not exist (becomes infinite).


Fig. 94

The graph shown in Fig. 94 is another instance of a function whose derivative does not vanish in the interval $[0,2]$.

The conditions of the Rolle theorem are not fulfilled for this function either, because at the point $x=1$ the function has no derivative.

## THE MEAN-VALUE THEOREM (LAGRANGE'S THEOREM)

Lagrange's Theorem. If a function $f(x)$ is continuous on the interval $[a, b]$ and differentiable at all interior points of the inter'val, there will be, within $[a, b]$, at least one point $c, a<c<b$, such that

$$
\begin{equation*}
f(b)-f(a)=f^{\prime}(c)(b-a) \tag{1}
\end{equation*}
$$

Proof. Let us denote by $Q$ the number $\frac{f(b)-f(a)}{b-a}$ that is, set:

$$
\begin{equation*}
Q=\frac{f(b)-f(a)}{b-a} \tag{2}
\end{equation*}
$$

und let us consider the auxiliary function $F(x)$ defined by the rquation

$$
\begin{equation*}
F(x)=f(x)-f(a)-(x-a) Q \tag{3}
\end{equation*}
$$

What is the geometric significance of the function $F(x)$ ? First write the equation of the chord $A B$ (Fig. 95), taking into account that its slope is $\frac{f(b)-f(a)}{b-a}=Q$ and lhat it passes through the point (a, $f(a))$ :

$$
y-f(a)=Q(x-a)
$$

whence

$$
y=f(a)+Q(x-a)
$$

But $\quad F(x)=f(x)-[f(a)+Q(x--a)]$. Thus, for each value of $x, F(x)$ is rqual to the difference between the ordinates of the curve $y=f(x)$ and the chord $y=f(a)+Q(x-a)$ for


Fig. 95 points with the same abscissa.

It will readily be seen that $F(x)$ is continuous on the interval $[a, b]$, is differentiable within the interval, and vanishes at the and points of the interval; in other words, $F(a)=0, F(b)=0$. llence, the Rolle theorem is applicable to the function $F(x)$. By this theorem, there exists within the interval a point $x=c$ such that

$$
F^{\prime}(c)=0
$$

But

$$
F^{\prime}(x)=f^{\prime}(x)-Q
$$

und so

$$
F^{\prime}(c)=f^{\prime}(c)-Q=0
$$

whence

$$
Q=f^{\prime}(c)
$$

Substituting the value of $Q$ in (2), we get

$$
\frac{f(b)-f(a)}{b-a}=f^{\prime}(c)
$$

whence follows formula (1) directly. The theorem is thus proved.
See Fig. 95 for an explanation of the geometric significance of the Lagrange theorem. From the figure it is immediately clear that the quantity $\frac{f(b)-f(a)}{b-a}$ is the tangent of the angle of inclination $\alpha$ of the chord passing through the points $A$ and $B$ with abscissas $a$ and $b$.
On the other hand, $f^{\prime}(c)$ is the tangent of the angle of inclination of the tangent line to the curve at the point with abscissa $c$. Thus, the geometric significance of ( $1^{\prime}$ ) or its equivalent (1) consists in the following: if at all points of the arc $A B$ there is a tangent line, then there will be, on this arc, a point $C$ between $A$ and $B$ at which the tangent is parallel to the chord connecting points $A$ and $B$.

Now note the following. Since the value of $c$ satisfies the condition $a<c<b$, it follows that $c-a<b-a$, or

$$
c-a=\theta(b-a)
$$

where $\theta$ is a certain number between 0 and 1 , that is,

$$
0<\theta<1
$$

But then

$$
c=a+\theta(b-a)
$$

and formula (1) may be written as follows:

$$
f(b)-f(a)=(b-a) f^{\prime}[a+\theta(b-a)], \quad 0<\theta<1
$$

## THE GENERALIZED MEAN-VALUE THEOREM (CAUCHY'S THEOREM)

Cauchy's Theorem. If $f(x)$ and $\varphi(x)$ are two functions continuous on an interval $[a, b]$ and differentiable within it, and $\varphi^{\prime}(x)$ does not vanish anywhere inside the interval, there will be, in $[a, b]$, $a$ point $x=c, a<c<b$, such that

$$
\begin{equation*}
\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f^{\prime}(c)}{\varphi^{\prime}(c)} \tag{1}
\end{equation*}
$$

Proof. Let us define the number $Q$ by the equation

$$
\begin{equation*}
Q=\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)} \tag{2}
\end{equation*}
$$

It will be noted that $\varphi(b)-\varphi(a) \neq 0$, since otherwise $\varphi(b)$ would be equal to $\varphi(a)$, and then, by the Rolle theorem, the derivative
$\varphi^{\prime}(x)$ would vanish in the interval; but this contradicts the statement of the theorem.

Let us construct an auxiliary function

$$
F(x)=f(x)-f(a)-Q[\varphi(x)-\varphi(a)]
$$

It is obvious that $F(a)=0$ and $F(b)=0$ (this follows from the definition of the function $F(x)$ and the definition of the number $Q$ ). Noting that the function $F(x)$ satisfies all the hypotheses of the Rolle theorem on the interval $[a, b]$, we conclude that there exists between $a$ and $b$ a value $x=c \quad(a<c<b)$ such that $F^{\prime}(c)=0$. But $F^{\prime}(x)=f^{\prime}(x)-Q \varphi^{\prime}(x)$, hence

$$
F^{\prime}(c)=f^{\prime}(c)-Q \varphi^{\prime}(c)=0
$$

whence

$$
Q=\frac{f^{\prime}(c)}{\varphi^{\prime}(c)}
$$

Substituting the value of $Q$ into (2) we get (1).
Note. The Cauchy theorem cannot be proved (as it might appear at first glance) by applying the Lagrange theorem to the numerator and denominator of the fraction

$$
\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}
$$

Indeed, in this case we would (after cancelling out $b-a$ ) get the formula

$$
\frac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\frac{f^{\prime}\left(c_{1}\right)}{\varphi^{\prime}\left(c_{2}\right)}
$$

in which $a<c_{1}<b, a<c_{2}<b$. But since, generally, $c_{1} \neq c_{2}$, the result obtained obviously does not yet yield the Cauchy theorem.
the limit of a ratio of two infinitesimals
(EVALUATING indeterminate forms of the type $\frac{\mathbf{0}}{\mathbf{0}}$ )
Let the functions $f(x)$ and $\varphi(x)$, on a certain interval $[a, b]$, satisfy the Cauchy theorem and vanish at the point $x=a$ of this interval, $f(a)=0$ and $\varphi(a)=0$.

The ratio $\frac{f(x)}{\varphi(x)}$ is not defined for $x=a$, but has a very definite meaning for values of $x \neq a$. Hence, we can raise the question of searching for the limit of this ratio as $x \rightarrow a$. Evaluating limits of this type is usually known as evaluating indeterminate forms of the type $\frac{0}{0}$.

We have already encountered such problems, for instance when considering the limit $\lim _{x \rightarrow 0} \frac{\sin x}{x}$ and when finding derivatives of elementary functions. For $x=0$, the expression $\frac{\sin x}{x}$ is meaningless; the function $F(x)=\frac{\sin x}{x}$ is not defined for $x=0$, but we have seen that the limit of the expression $\frac{\sin x}{x}$ as $x \rightarrow 0$ exists and is equal to unity.

L'Hospital's Theorem (Rule). Let the functions $f(x)$ and $\varphi(x)$, in $[a, b]$, satisfy the Cauchy theorem and vanish at the point $x=a$, that is, $f(a)=\varphi(a)=0$; then, if the ratio $\frac{f^{\prime}(x)}{\varphi^{\prime}(x)}$ has a limit as $x \rightarrow a$, there also exists $\lim _{x \rightarrow a} \frac{f(x)}{\varphi(x)}$, and

$$
\lim _{x \rightarrow a} \frac{f(x)}{\varphi(x)}=\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}
$$

Proof. On the interval $[a, b]$ take some point $x \neq a$. Applying the Cauchy formula we have

$$
\frac{f(x)-f(x)}{\varphi(x)-\varphi(a)}=\frac{f^{\prime}(\xi)}{\varphi^{\prime}(\xi)}
$$

where $\xi$ lies between $a$ and $x$. But it is given that $f(a)=\varphi(a)=0$, and so

$$
\begin{equation*}
\frac{f(x)}{\varphi(x)}=\frac{f^{\prime}(\xi)}{\varphi^{\prime}(\xi)} \tag{1}
\end{equation*}
$$

If $x \rightarrow a$, then $\xi \rightarrow a$ also, since $\xi$ lies between $x$ and $a$. And if $\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}=A$, then $\lim _{\xi \rightarrow a} \frac{f^{\prime}(\xi)}{\varphi^{\prime}(\xi)}$ exists and is equal to $A$. Whence it is clear that

$$
\lim _{x \rightarrow a} \frac{f(x)}{\varphi(x)}=\lim _{x \rightarrow a} \frac{f^{\prime}(\xi)}{\varphi^{\prime}(\xi)}=\lim _{\xi \rightarrow a} \frac{f^{\prime}(\xi)}{\varphi^{\prime}(\xi)}=\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}=A
$$

and, finally,

$$
\lim _{x \rightarrow a} \frac{f(x)}{\varphi(x)}=\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}
$$

Note 1. The theorem holds also for the case where the functions $f(x)$ or $\varphi(x)$ are not defined at $x=a$, but

$$
\lim _{x \rightarrow a} f(x)=0, \quad \lim _{x \rightarrow a} \varphi(x)=0
$$

In order to reduce this case to the earlier considered case, we redefine the functions $f(x)$ and $\varphi(x)$ at the point $x=a$ so that
lhey become continuous at the point $a$. To do this, it is sufficient to put

$$
f(a)=\lim _{x \rightarrow a} f(x)=0, \quad \varphi(a)=\lim _{x \rightarrow a} \varphi(x)=0
$$

since it is obvious that the limit of the ratio $\frac{f(x)}{\varphi(x)}$ as $x \rightarrow a$ does not depend on whether the functions $f(x)$ and $\varphi(x)$ are defined "1 $x=a$.

Note 2. If $f^{\prime}(a)=\varphi^{\prime}(a)=0$ and the derivatives $f^{\prime}(x)$ and $\varphi^{\prime}(x)$ antisfy the conditions that were imposed by the theorem on the functions $f(x)$ and $\varphi(x)$, then applying the l'Hospital rule to the ratio $\frac{f^{\prime}(x)}{\varphi^{\prime}(x)}$, we arrive at the formula $\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}=\lim _{x \rightarrow a} \frac{f^{\prime \prime}(x)}{\varphi^{\prime \prime}(x)}$, and s) forth.

Note 3. If $\varphi^{\prime}(a)=0$, but $f^{\prime}(x) \neq 0$, then the theorem is applicable to the reciprocal ratio $\frac{\varphi(x)}{f(x)}$, which tends to zero as $x \rightarrow a$. Hence, the ratio $\frac{f(x)}{\varphi(x)}$ tends to infinity.

## Example 1.

$$
\lim _{x \rightarrow 0} \frac{\sin 5 x}{3 x}=\lim _{x \rightarrow 0} \frac{(\sin 5 x)^{\prime}}{(3 r)^{\prime}}=\lim _{x \rightarrow 0} \frac{5 \cos 5 x}{3}=\frac{5}{3}
$$

## Example 2.

$$
\lim _{x \rightarrow 0} \frac{\ln (1+x)}{x}=\lim _{x \rightarrow 0} \frac{\frac{1}{1+x}}{1}=\frac{1}{1}=1
$$

## Example 3.

$\lim _{1 \rightarrow 0} \frac{e^{x}-e^{-x}-2 x}{x-\sin x}=\lim _{x \rightarrow 0} \frac{e^{x}+e^{-x}-2}{1-\cos x}=\lim _{x \rightarrow 0} \frac{e^{x}-e^{-x}}{\sin x}=\lim _{x \rightarrow 0} \frac{e^{x}+e^{-x}}{\cos x}=\frac{2}{1}=2$
Here, we had to apply the l'Hospital rule three times because the ratios of the first, second and third derivatives at $x=0$ yield the indeterminate form $\frac{0}{0}$.

Note 4. The l'Hospital rule is also applicable if

$$
\lim _{x \rightarrow \infty} f(x)=0 \quad \text { and } \quad \lim _{x \rightarrow \infty} \varphi(x)=0
$$

Indeed, putting $x=\frac{1}{z}$, we see that $z \rightarrow 0$ as $x \rightarrow \infty$ and therefor $\epsilon$

$$
\lim _{z \rightarrow 0} f\left(\frac{1}{z}\right)=0, \quad \lim _{z \rightarrow 0} \varphi\left(\frac{1}{z}\right)=0
$$

Applying the l'Hospital rule to the ratio $\frac{f\left(\frac{1}{z}\right)}{\varphi\left(\frac{1}{z}\right)}$, we find

$$
\begin{aligned}
\lim _{x \rightarrow \infty} \frac{f(x)}{\varphi(x)} & =\lim _{z \rightarrow 0} \frac{f\left(\frac{1}{z}\right)}{\varphi\left(\frac{1}{z}\right)}=\lim _{z \rightarrow 0} \frac{f^{\prime}\left(\frac{1}{z}\right)\left(-\frac{1}{z^{2}}\right)}{\varphi^{\prime}\left(\frac{1}{z}\right)\left(-\frac{1}{z^{2}}\right)} \\
& =\lim _{z \rightarrow 0} \frac{f^{\prime}\left(\frac{1}{z}\right)}{\varphi^{\prime}\left(\frac{1}{z}\right)}=\lim _{x \rightarrow \infty} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}
\end{aligned}
$$

which is what we wanted to prove.
Example 4.

$$
\lim _{x \rightarrow \infty} \frac{\sin \frac{k}{x}}{\frac{1}{x}}=\lim _{x \rightarrow \infty} \frac{k \cos \frac{k}{x}\left(-\frac{1}{x^{2}}\right)}{-\frac{1}{x^{2}}}=\lim _{x \rightarrow \infty} k \cos \frac{k}{x}=k
$$

## the limit of a ratio of two infinitely LARGE QUANTITIES

## (EVALUATING indeterminate forms of the type $\frac{\infty}{\infty}$ )

Let us now consider the question of the limit of a ratio of two functions $f(x)$ and $\varphi(x)$ approaching infinity as $x \rightarrow a$ (or as $x \rightarrow \infty$ ).

Theorem. Let the functions $f(x)$ and $\varphi(x)$ be continuous and differentiable for all $x \neq a$ in the neighbourhood of the point $a$, the derivative $\varphi^{\prime}(x)$ does not vanish; further, let

$$
\lim _{x \rightarrow a} f(x)=\infty, \quad \lim _{x \rightarrow a} \varphi(x)=\infty
$$

and let there.be a limit

$$
\begin{equation*}
\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}=A \tag{1}
\end{equation*}
$$

Then there is a limit $\lim _{x \rightarrow a} \frac{f(x)}{\varphi(x)}$ and

$$
\begin{equation*}
\lim _{x \rightarrow a} \frac{f(x)}{\varphi(x)}=\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}=A \tag{2}
\end{equation*}
$$

Proof. In the given neighbourhood of the point $a$, take two points $\alpha$ and $x$ such that $\alpha<x<a$ (or $a<x<\alpha$ ). By Cauchy's theorem we have

$$
\begin{equation*}
\frac{f(x)-f(\alpha)}{\varphi(x)-\varphi(\alpha)}=\frac{f^{\prime}(c)}{\varphi^{\prime}(c)} \tag{3}
\end{equation*}
$$

where $\alpha<c<x$. We transform the left side of (3) as follows:

$$
\begin{equation*}
\frac{f(x)-f(\alpha)}{\varphi(x)-\varphi(\alpha)}=\frac{f(x)}{\varphi(x)} \frac{1-\frac{f(\alpha)}{P(x)}}{1-\frac{\varphi(\alpha)}{\varphi(x)}} \tag{4}
\end{equation*}
$$

From relations (3) and (4) we have

$$
\frac{f^{\prime}(c)}{\varphi^{\prime}(c)}=\frac{f(x)}{\varphi(x)} \frac{1-\frac{f(\alpha)}{f(x)}}{1-\frac{\varphi(\alpha)}{\varphi(x)}}
$$

Whence we find

$$
\begin{equation*}
\frac{f(x)}{\varphi(x)}=\frac{f^{\prime}(c)}{\varphi^{\prime}(c)} \frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}} \tag{5}
\end{equation*}
$$

From condition (1) it follows that for an arbitrarily small $\varepsilon>0$, $x$ may be chosen so close to $a$ that for all $x=c$ where $\alpha<c<a$, the following inequality will be fulfilled:

$$
\left|\frac{f^{\prime}(c)}{\varphi^{\prime}(c)}-A\right|<\varepsilon
$$

or

$$
\begin{equation*}
A-\varepsilon<\frac{f^{\prime}(c)}{\varphi^{\prime}(c)}<A+\varepsilon \tag{6}
\end{equation*}
$$

l.et us further consider the fraction

$$
\frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}}
$$

Fixing $\alpha$ so that inequality (6) holds, we allow $x$ to approach $a$. Since $f(x) \longrightarrow \infty$ and $\varphi(x) \longrightarrow \infty$ as $x \rightarrow a$, we have

$$
\lim _{x \rightarrow a} \frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}}=1
$$

ind, consequently, for the earlier chosen $\varepsilon>0$ (for $x$ sufficiently close to $a$ ) we will have

$$
\left|\frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}}-1\right|<\varepsilon
$$

or

$$
\begin{equation*}
1-\varepsilon<\frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}}<1+\varepsilon \tag{7}
\end{equation*}
$$

Multiplying together the appropriate terms of inequalities (6) and (7), we get

$$
(A-\varepsilon)(1-\varepsilon)<\frac{f^{\prime}(c)}{\varphi^{\prime}(c)} \frac{1-\frac{\varphi(\alpha)}{\varphi(x)}}{1-\frac{f(\alpha)}{f(x)}}<(A+\varepsilon)(1+\varepsilon)
$$

or, from (5),

$$
(A-\varepsilon)(1-\varepsilon)<\frac{f(x)}{\varphi(x)}<(A+\varepsilon)(1+\varepsilon)
$$

Since $\varepsilon$ is an arbitrarily small number for $x$ sufficiently close to $a$, it follows from the latter inequalities that

$$
\lim _{x \rightarrow a} \frac{f(x)}{\varphi(x)}=A
$$

or, by (1),

$$
\lim _{x \rightarrow a} \frac{f(x)}{\varphi(x)}=\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}=A
$$

which completes the proof.
Note 1. If in condition (1) $A=\infty$, that is,

$$
\lim _{x \rightarrow a} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}=\infty
$$

then (2) holds in this case as well. Indeed, from the preceding expression it follows that

$$
\lim _{x \rightarrow a} \frac{\varphi^{\prime}(x)}{f^{\prime}(x)}=0
$$

Then by the theorem just proved

$$
\lim _{x \rightarrow a} \frac{\varphi(x)}{f(x)}=\lim _{x \rightarrow a} \frac{\varphi^{\prime}(x)}{f^{\prime}(x)}=0
$$

whence

$$
\lim _{x \rightarrow a} \frac{f(x)}{\varphi(x)}=\infty
$$

Note 2. The theorem just proved is readily extended to the case where $x \rightarrow \infty$. If $\lim _{x \rightarrow \infty} f(x)=\infty, \lim _{x \rightarrow \infty} \varphi(x)=\infty$ and $\lim _{x \rightarrow \infty} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)}$
exists, then

$$
\begin{equation*}
\lim _{x \rightarrow \infty} \frac{f(x)}{\varphi(x)}=\lim _{x \rightarrow \infty} \frac{f^{\prime}(x)}{\varphi^{\prime}(x)} \tag{8}
\end{equation*}
$$

The proof is carried out by the substitution $x=\frac{1}{2}$, as was done under similar conditions in the case of the indeterminate form ${ }_{0}^{0}$ (see Sec. 4.4, Note 4).

## Example 1.

$$
\lim _{x \rightarrow \infty} \frac{e^{x}}{x}=\lim _{x \rightarrow \infty} \frac{\left(e^{x}\right)^{\prime}}{(x)^{\prime}}=\lim _{x \rightarrow \infty} \frac{e^{x}}{1}=\infty
$$

Note 3. Once again note that formulas (2) and (8) hold only if the limit on the right (finite or infinite) exists. It may happen that the limit on the left exists while there is no limit on the right. To illustrate, let it be required to find

$$
\lim _{x \rightarrow \infty} \frac{x+\sin x}{x}
$$

This limit exists and is equal to 1 . Indeed,

$$
\lim _{x \rightarrow \infty} \frac{x+\sin x}{x}=\lim _{x \rightarrow \infty}\left(1+\frac{\sin x}{x}\right)=1
$$

But the ratio of derivatives

$$
\frac{(x+\sin x)^{\prime}}{\left(x^{\prime}\right)}=\frac{1+\cos x}{1}=1+\cos x
$$

as $x \rightarrow \infty$ does not approach any limit, it oscillates between 0 and 2.

## Example 2.

$$
\lim _{x \rightarrow \infty} \frac{a x^{2}+b}{c x^{2}-d}=\lim _{x \rightarrow \infty} \frac{2 a x}{2 c x}=\frac{a}{c}
$$

## Example 3.

$$
\begin{aligned}
& \lim _{x \rightarrow \frac{\pi}{2}} \frac{\tan x}{\tan 3 x}=\lim _{x \rightarrow \frac{\pi}{2}} \frac{\frac{1}{\cos ^{2} x}}{\frac{3}{\cos ^{2} 3 x}}=\lim _{x \rightarrow \frac{\pi}{2}} \frac{1}{3} \frac{\cos ^{2} 3 x}{\cos ^{2} x}=\lim _{x \rightarrow \frac{\pi}{2}} \frac{1}{3} \frac{2 \cdot 3 \cos 3 x \sin 3 x}{2 \cos x \sin x} \\
& \quad=\lim _{x \rightarrow \frac{\pi}{2}} \frac{\cos 3 x}{\cos x} \lim _{x \rightarrow \frac{\pi}{2}} \frac{\sin 3 x}{\sin x}=\lim _{x \rightarrow \frac{\pi}{2}} \frac{3 \sin 3 x}{\sin x} \cdot \frac{(-1)}{(1)}=3 \frac{(-1)}{(1)} \cdot \frac{(-1)}{(1)}=3
\end{aligned}
$$

## Example 4.

$$
\lim _{x \rightarrow \infty} \frac{x}{e^{x}}=\lim _{x \rightarrow \infty} \frac{1}{e^{x}}=0
$$

Generally, for any integral $n>0$,

$$
\lim _{x \rightarrow \infty} \frac{x^{n}}{e^{x}}=\lim _{x \rightarrow \infty} \frac{n x^{n-1}}{e^{x}}=\ldots=\lim _{x \rightarrow \infty} \frac{n(n-1) \ldots 1}{e^{x}}=0
$$

The other indeterminate forms reduce to the foregoing cases. These forms may be written symbolically as follows:
(a) $0 . \infty$,
(b) $0^{0}$,
(c) $\infty^{0}$,
(d) $1^{\infty}$,
(e) $\infty-\infty$

They have the following meaning.
(a) Let $\lim _{x \rightarrow a} f(x)=0 ; \lim _{x \rightarrow a} \varphi(x)=\infty$; it is required to find

$$
\lim _{x \rightarrow a}[f(x) \varphi(x)]
$$

that is, the indeterminate form $0 . \infty$.
If the required expression is rewritten as follows:

$$
\lim _{x \rightarrow a}[f(x) \varphi(x)]=\lim _{x \rightarrow a} \frac{f(x)}{\frac{1}{\varphi(x)}}
$$

or in the form

$$
\lim _{x \rightarrow a}[f(x) \varphi(x)]=\lim _{x \rightarrow a} \frac{\varphi(x)}{\frac{1}{f(x)}}
$$

then as $x \rightarrow a$ we obtain the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

## Example 5.

$$
\lim _{x \rightarrow 0} x^{n} \ln x=\lim _{x \rightarrow 0} \frac{\ln x}{\frac{1}{x^{n}}}=\lim _{x \rightarrow 0} \frac{\frac{1}{x}}{-\frac{n}{x^{n+1}}}=-\lim _{x \rightarrow 0} \frac{x^{n}}{n}==0
$$

(b) Let

$$
\lim _{x \rightarrow a} f(x)=0, \lim _{x \rightarrow a} \varphi(x)=0
$$

it is required to find

$$
\lim _{x \rightarrow a}[f(x)]^{\tilde{(x}}
$$

or, as we say, to evaluate the indeterminate form $0^{0}$.
Putting

$$
y=[f(x)]^{\dot{\beta}}(x)
$$

lake logarithms of both sides of the equation:

$$
\ln y=\varphi(x)[\ln f(x)]
$$

As $x \rightarrow a$ we obtain (on the right) the indeterminate form $0 \cdot \infty$. 1 inding $\lim _{x \rightarrow a} \ln y$, it is easy to get $\lim _{x \rightarrow a} y$. Indeed, by virtue of the
 In $\lim _{x \rightarrow a} y=b$, it is obvious that $\lim _{x \rightarrow a} y=e^{\boldsymbol{b}}$. If, in particular, $b=+\infty$ or $\stackrel{x \rightarrow a}{ }-\infty$, then we will have $\lim \stackrel{x \rightarrow a}{y}=+\infty$ or 0 , respectively.

Example 6. It is required to find $\lim _{x \rightarrow 0} x^{x}$. Putting $y=x^{x}$ we find $\ln \lim y=$ $-\lim \ln y=\lim \ln \left(x^{x}\right)=\lim (x \ln x)$;

$$
\lim _{x \rightarrow 0}(x \ln x)=\lim _{x \rightarrow 0} \frac{\ln x}{\frac{1}{x}}=\lim _{x \rightarrow 0} \frac{\frac{1}{x}}{-\frac{1}{x^{2}}}=-\lim _{x \rightarrow 0} x=0
$$

Consequently, $\ln \lim y=0$, whence $\lim y=e^{0}=1$, or

$$
\lim _{x \rightarrow 0} x^{x}=1
$$

The technique is similar for finding limits in other cases.

## TAYLOR'S FORMULA

Let us assume that the function $y=f(x)$ has all derivatives up to the $(n+1$ )th order, inclusive, in some interval containing the point $x=a$. Let us find a polynomial $y=P_{n}(x)$ of degree not ubove $n$, the value of which at $x=a$ is equal to the value of the lunction $f(x)$ at this point, and the values of its derivatives up (o) the $n$th order at $x=a$ are equal to the values of the correspondling derivatives of the function $f(x)$ at this point:
$P_{n}(a)=f(a), P_{n}^{\prime}(a)=f^{\prime}(a), P_{n}^{\prime \prime}(a)=f^{\prime \prime}(a), \ldots, P_{n}^{(n)}(a)=f^{(n)}(a)$
It is natural to expect that, in a certain sense, such a polynomial Is "close" to the function $f(x)$.

Let us look for this polynomial in the form of a polynomial in powers of $(x-a)$ with undetermined coefficients:

$$
\begin{gather*}
P_{n}(x)=C_{0}+C_{1}(x-a)+C_{2}(x-a)^{2}+C_{3}(x-a)^{3} \\
+\ldots+C_{n}(x-a)^{n} \tag{2}
\end{gather*}
$$

We define the undetermined coefficients $C_{1}, C_{2}, \ldots, C_{n}$ so that conditions (1) are satisfied,

Let us first find the derivatives of $P_{n}(x)$ :

Substituting, into the left and right sides of (2) and (3), the value of $a$ in place of $x$ and replacing, by (1), $P_{n}(a)$ by $f(a)$, $P_{n}^{\prime}(a)=f^{\prime}(a)$, etc., we get

$$
\begin{aligned}
f(a) & =C_{0} \\
f^{\prime}(a) & =C_{1} \\
f^{\prime \prime}(a) & =2 \cdot 1 C_{2} \\
f^{\prime \prime \prime}(a) & =3 \cdot 2 \cdot 1 C_{3} \\
f^{(n)}(a) & =\dot{n}(\dot{n}-1)(\dot{n}-2) \ldots 2 \cdot 1 C_{n}
\end{aligned}
$$

whence we find

$$
\left.\begin{array}{l}
C_{0}=f(a), \quad C_{1}=f^{\prime}(a), \quad C_{2}=\frac{1}{1 \cdot 2} f^{\prime \prime}(a),  \tag{4}\\
C_{3}=\frac{1}{1 \cdot 2 \cdot 3} f^{\prime \prime \prime}(a), \quad \ldots, \quad C_{n}=\frac{1}{1 \cdot 2 \quad . n} f^{(n)}(a)
\end{array}\right\}
$$

Substituting into (2) the values of $C_{1}, C_{2}, \ldots, C_{n}$ that have been found, we get the required polynomial:

$$
\begin{align*}
P_{n}(x)=f(a)+\frac{x-a}{1} f^{\prime}(a)+\frac{(x-a)^{2}}{1 \cdot 2} f^{\prime \prime}(a) & +\frac{(x-a)^{3}}{1 \cdot 2 \cdot 3} f^{\prime \prime \prime}(a) \\
& +\ldots+\frac{(x-a)^{n}}{1 \cdot 2 . . n} f^{(n)}(a) \tag{5}
\end{align*}
$$

Designate by $R_{n}(x)$ the difference between the values of the given function $f(x)$ and the constructed polynomial $P_{n}(x)$ (Fig. 96):

$$
R_{n}(x)=f(x)-P_{n}(x)
$$

whence

$$
f(x)=P_{n}(x)+R_{n}(x)
$$

or, in expanded form,

$$
\begin{align*}
f(x)=f(a)+\frac{x-a}{1!} f^{\prime}(a) & +\frac{(x-a)^{2}}{2!} f^{\prime \prime}(a) \\
& +\ldots+\frac{(x-a)^{n}}{n!} f^{(n)}(a)+R_{n}(x) \tag{6}
\end{align*}
$$

$R_{n}(x)$ is called the remainder. For those values of $x$, for which the remainder $R_{n}(x)$ is small, the polynomial $P_{n}(x)$ yields an approximate representation of the function $f(x)$.

Thus, formula (6) enables one to replace the function $y=f(x)$ by the polynomial $y=P_{n}(x)$ to an appropriate degree of accuracy equal to the value of the remainder $R_{n}(x)$.
Our next problem is to evaluate the quantity $R_{n}(x)$ for various values of $x$.

Let us write the remainder in the form


Fig. 96

$$
\begin{equation*}
R_{n}(x)=\frac{(x-a)^{n+1}}{(n+1)!} Q(x) \tag{7}
\end{equation*}
$$

where $Q(x)$ is a certain function to be defined, and accordingly rewrite (6):

$$
\begin{align*}
f(x)=f(a)+\frac{x-a}{1} f^{\prime}(a) & +\frac{(x-a)^{2}}{2!} f^{\prime \prime}(a) \\
& +\ldots+\frac{(x-a)^{n}}{n!} f^{(n)}(a)+\frac{(x-a)^{n+1}}{(n+1)!} Q(x)
\end{align*}
$$

For fixed $x$ and $a$, the function $Q(x)$ has a definite value; denote It by $Q$.

Let us further examine the auxiliary function of $t$ ( lying between $a$ and $x$ ):

$$
\begin{aligned}
\because(t)=f(x)-f(t)-\frac{x-t}{1} f^{\prime}(t)-\frac{(x-t)^{2}}{2!} & f^{\prime \prime}(t)-\ldots \\
& -\frac{(x-t)^{n}}{n!} f^{(n)}(t)-\frac{(x-t)^{n+1}}{(n+1)!} Q
\end{aligned}
$$

where $Q$ has the value defined by the relation ( $6^{\prime}$ ); here we consider $a$ and $x$ to be definite numbers.

We find the derivative $F^{\prime}(t)$ :

$$
\begin{aligned}
& f^{\prime \prime}(t)=-f^{\prime}(t)+f^{\prime}(t)-\frac{x-t}{1} f^{\prime \prime}(t)+\frac{2(x-t)}{2!} f^{\prime \prime}(t) \\
& -\frac{(x-t)^{2}}{2!} f^{\prime \prime \prime}(t)+\ldots+\frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)+\frac{n(x-t)^{n-1}}{n!} f^{(n)}(t) \\
& \quad-\frac{(x-t)^{n}}{n!} f^{(n+1)}(t)+\frac{(n+1)(x-t)^{n}}{(n+1)!} Q
\end{aligned}
$$

or, on cancelling,

$$
\begin{equation*}
F^{\prime}(t)=-\frac{(x-t)^{n}}{n!} f^{(n+1)}(t)+\frac{(x-t)^{n}}{n!} Q \tag{8}
\end{equation*}
$$

Thus, the function $F(t)$ has a derivative at all points $t$ lying near the point with abscissa $a(a \leqslant t \leqslant x$ when $a<x$ and $a \geqslant t \geqslant x$ when $a>x$ ).

It will further be noted that, on the basis of $\left(6^{\prime}\right)$,

$$
F(x)=0, \quad F(a)=0
$$

Therefore, the Rolle theorem is applicable to the function $F(t)$ and, consequently, there exists a value $t=\xi$ lying between $a$ and $x$ such that $F^{\prime}(\xi)=0$. Whence, on the basis of relation (8), we get

$$
-\frac{(x-\xi)^{n}}{n!} f^{(n+1)}(\xi)+\frac{(x-\xi)^{n}}{n!} Q=0
$$

and from this

$$
Q=f^{(n+1)}(\xi)
$$

Substituting this expression into (7), we get

$$
R_{n}(x)=\frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)
$$

This is the so-called Lagrange form of the remainder.
Since $\xi$ lies between $x$ and $a$, it may be represented in the form

$$
\xi=a+\theta(x-a)
$$

where $\theta$ is a number lying between 0 and 1 , that is, $0<\theta<1$; then the formula of the remainder takes the form

$$
R_{n}(x)=\frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}[a+\theta(x-a)]
$$

The formula

$$
\begin{align*}
f(x)=f(a)+ & \frac{x-a}{1!} f^{\prime}(a)+\frac{(x-a)^{2}}{2!} f^{\prime \prime}(a)+\ldots \\
& +\frac{(x-a)^{n}}{n!} f^{(n)}(a)+\frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}[a+\theta(x-a)] \tag{9}
\end{align*}
$$

is called Taylor's formula of the function $f(x)$.
If in the Taylor formula we put $a=0$ we will have

$$
\begin{align*}
f(x)=f(0)+\frac{x}{1!} f^{\prime}(0)+\frac{x^{2}}{2!} f^{\prime \prime}(0) & +\ldots \\
& +\frac{x^{n}}{n!} f^{(n)}(0)+\frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x) \tag{10}
\end{align*}
$$

where $\theta$ lies between 0 and 1 . This special case of the Taylor formula is sometimes called Maclaurin's formula.

## EXPANSION OF THE FUNCTIONS $e^{\boldsymbol{x}}, \sin x$, AND $\cos x$ IN A TAYLOR SERIES

1. Expansion of the function $\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{e}^{\boldsymbol{x}}$. Finding the successive derivatives of $f(x)$, we have

$$
\begin{gathered}
f(x)=e^{x}, \quad f(0)=1 \\
f^{\prime}(x)=e^{x}, \quad f^{\prime}(0)=1 \\
\cdot \quad . \quad . \quad . \quad .
\end{gathered}
$$

Substituting the expressions obtained into formula (10) (Sec. 4.6), we get

$$
e^{x}=1+\frac{x}{11}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\ldots+\frac{x^{n}}{n!}+\frac{x^{n+1}}{(n+1)!} e^{\theta x}, \quad 0<\theta<1
$$

If $|x| \leqslant 1$, then, taking $n=8$, we obtain an estimate of the remainder:

$$
R_{8}<\frac{1}{9!} 3<10^{-5}
$$

For $x=1$ we get a formula that permits approximating the number $e$ :

$$
e=1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{8!}
$$

Calculating to the sixth decimal place,* and then rounding to five decimals, we have

$$
e=2.71828
$$

llere the error does not exceed $\frac{3}{91}$, or 0.00001 .
Observe that no matter what $x$ is, the remainder

$$
R_{n}=\frac{x^{n+1}}{(n+1)!} e^{\theta x} \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

Indeed, since $\theta<1$, the quantity $e^{9 x}$ for fixed $x$ is bounded (it is less than $e^{x}$ for $x>0$ and less than 1 for $x<0$ ).

We shall prove that, no matter what the fixed number $x$,

$$
\frac{x^{n+1}}{(n+1)!} \rightarrow 0 \quad \text { as } \quad n \rightarrow \infty
$$

Indeed,

$$
\left|\frac{x^{n+1}}{(n+1)!}\right|=\left|\frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \ldots \frac{x}{n} \cdot \frac{x}{n+1}\right|
$$

[^1]If $x$ is a fixed number, there will be a positive integer $N$ such that

$$
|x|<N
$$

We introduce the notation $\frac{|x|}{N}=q$; then, noting that $0<q<1$, we can write for $n=N+1, N+2, N+3$, etc.

$$
\begin{gathered}
\left|\frac{x^{n+1}}{(n+1)!}\right|=\left|\frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{n} \cdot \frac{x}{n+1}\right| \\
=\left|\frac{x}{1}\right|\left|\frac{x}{2}\right|\left|\frac{x}{3}\right| \cdots\left|\frac{x}{N-1}\right| \cdot\left|\frac{x}{N}\right| \cdots\left|\frac{x}{n}\right| \cdot\left|\frac{x}{n+1}\right| \\
<\frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{N-1} \cdot q \cdot q \ldots q=\frac{x^{N-1}}{(N-1)!} q^{n-N+2}
\end{gathered}
$$

for the reason that

$$
\left|\frac{x}{N}\right|=q, \quad\left|\frac{x}{N+1}\right|<q, \ldots,\left|\frac{x}{n+1}\right|<q
$$

But $\frac{x^{N-1}}{(N-1)!}$ is a constant quantity; that is to say, it is independent of $n$, while $q^{n-N+2}$ approaches zero as $n \rightarrow \infty$. And so

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!}=0 \tag{1}
\end{equation*}
$$

Consequently, $R_{n}(x)=e^{\theta x} \frac{x^{n+1}}{(n+1)!}$ also approaches zero as $n$ approaches infinity.

From the foregoing it follows that for any $x$ (if a sufficient number of terms is taken) we can evaluate $e^{x}$ to any degree of accuracy.
2. Expansion of the function $f(x)=\sin x$. We find the successive derivatives of $f(x)=\sin x$ :

$$
\begin{aligned}
& f(x)=\sin x \\
& f(0)=0 \\
& f^{\prime}(x)=\cos x=\sin \left(x+\frac{\pi}{2}\right), \\
& f^{\prime}(0)=1 \\
& f^{\prime \prime}(x)=-\sin x=\sin \left(x+2 \frac{\pi}{2}\right), \\
& f^{\prime \prime}(0)=0 \\
& f^{\prime \prime \prime}(x)=-\cos x=\sin \left(x+3 \frac{\pi}{2}\right), \quad f^{\prime \prime \prime}(0)=-1 \\
& f^{\text {IV }}(x)=\sin x=\sin \left(x+4 \frac{\pi}{2}\right), \quad \quad f^{\text {IV }}(0)=0 \text {. } \\
& f^{(n)}(x)=\sin \left(x+n \frac{\pi}{2}\right), \\
& f^{(n)}(0)=\sin n \frac{\pi}{2} \\
& f^{(n+1)}(x)=\sin \left(x+(n+1) \frac{\pi}{2}\right), \quad f^{(n+1)}(\xi)=\sin \left[\xi+(n+1) \frac{\pi}{2}\right]
\end{aligned}
$$

Substituting the values obtained into (10), Sec. 4.6, we get an expansion of the function $f(x)=\sin x$ by the Taylor formula:
$\sin x=x-\frac{x^{3}}{3!}+\frac{x^{6}}{5!}-\ldots+\frac{x^{n}}{n!} \sin n \frac{\pi}{2}+\frac{x^{n+1}}{(n+1)!} \sin \left[\xi+(n+1) \frac{\pi}{2}\right]$
Since $\left|\sin \left[\xi+(n+1) \frac{\pi}{2}\right]\right| \leqslant 1$, we have $\lim _{n \rightarrow \infty} R_{n}(x)=0$ for all values of $x$.

Let us apply the formula in order to approximate $\sin 20^{\circ}$. Put " -3 , thus restricting ourselves to the first two terms of the expansion:

$$
\sin 20^{\circ}=\sin \frac{\pi}{9} \approx \frac{\pi}{9}-\frac{1}{3!}\left(\frac{\pi}{9}\right)^{3}=0.342
$$

Let us estimate the error, which is equal to the remainder:

$$
\left|R_{3}\right|=\left|\left(\frac{\pi}{9}\right)^{4} \frac{1}{4!} \sin (\xi+2 \pi)\right| \leqslant\left(\frac{\pi}{9}\right)^{4} \frac{1}{4!} \approx 0.00062<0.001
$$

Hence, the error is less than 0.001 , and so $\sin 20^{\circ}=0.342$ to lliree places of decimals.


Fig. 97
Fig. 97 shows the graphs of the function $f(x)=\sin x$ and the first three approximations: $S_{1}(x)=x, S_{2}(x)=x-\frac{x^{3}}{31}, S_{3}(x)=x-$ $-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}$.
3. Expansion of the function $f(x)=\cos \boldsymbol{x}$. Finding the values of the successive derivatives for $x=0$ of the function $f(x)=\cos x$
and substituting them into the Maclaurin formula, we get the expansion

$$
\begin{gathered}
\cos x=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\ldots+\frac{x^{n}}{n!} \cos \left(n \frac{\pi}{2}\right)+\frac{x^{n+1}}{(n+1)!} \cos \left[\xi+(n+1) \frac{\pi}{2}\right], \\
|\xi|<|x|
\end{gathered}
$$

Here again, $\lim _{n \rightarrow \infty} R_{n}(x)=0$ for all values of $x$.


[^0]:    - The number $c$ is called a root of the function $\varphi(x)$ if $\varphi(c)=0$.

[^1]:    * Otherwise the overall rounding error may considerably exceed $R_{B}$ (for Instance, for 10 terms, this error can exceed $5 \cdot 10^{-5}$ ).

