Lecture 10

DERIVATIVES OF DIFFERENT ORDERS

Let a function $y=f(x)$ be differentiable on some interval $[a, b]$. Generally speaking, the values of the derivative $f'(x)$ depend on x, which is to say that **the derivative $f'(x)$ is also a function of x.** Differentiating this function, we obtain the so-called second derivative of the function $f(x)$.

The derivative of a first derivative is called a *derivative of the second order* or the *second derivative* of the original function and is denoted by the symbol y'' or $f''(x)$:

$$y''=(y')'=f''(x)$$

For example, if $y=x^5$, then

$$y'=5x^4, \quad y''=(5x^4)'=20x^3$$

The derivative of the second derivative is called a *derivative of the third order* or the *third derivative* and is denoted by y''' or $f'''(x)$.

Generally, a *derivative of the nth order of a function* $f(x)$ is called the derivative (first-order) of the derivative of the $(n-1)$th order and is denoted by the symbol $y^{(n)}$ or $f^{(n)}(x)$:

$$y^{(n)}=(y^{(n-1)})'=f^{(n)}(x)$$

(The order of the derivative is taken in parentheses so as to avoid confusion with the exponent of a power.)

Derivatives of the fourth, fifth, and higher orders are also denoted by Roman numerals: y^{IV}, y^{V}, $y^{\mathrm{VI}}, \ldots$. Here, the order of the derivative may be written without brackets. For instance, if $y = x^5$, then $y' = 5x^4$, $y'' = 20x^3$, $y''' = 60x^2$, $y^{\mathrm{IV}} = y^{(4)} = 120x$, $y^{\mathrm{V}} = y^{(5)} = 120$, $y^{(6)} = y^{(7)} = \ldots = 0$.

Example 1. Given a function $y = e^{kx}$ ($k = \text{const}$). Find the expression of its derivative of any order n.

Solution. $y' = ke^{kx}$, $y'' = k^2e^{kx}$, $\ldots$, $y^{(n)} = k^ne^{kx}$.

Example 2. $y = \sin x$. Find $y^{(n)}$.

Solution.

$$y' = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

$$y'' = -\sin x = \sin\left(x + 2\frac{\pi}{2}\right)$$

$$y''' = -\cos x = \sin\left(x + 3\frac{\pi}{2}\right)$$

$$y^{\mathrm{IV}} = \sin x = \sin\left(x + 4\frac{\pi}{2}\right)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$y^{(n)} = \sin\left(x + n\frac{\pi}{2}\right)$$

In similar fashion we can also derive the formulas for the derivatives of any order of certain other elementary functions. The reader himself can find the formulas for derivatives of the nth order of the functions $y = x^k$, $y = \cos x$, $y = \ln x$.

The rules given in theorems 2 and 3, Sec. 3.7, are readily generalized to the case of derivatives of any order.

In this case we have obvious formulas:

$$(u + v)^{(n)} = u^{(n)} + v^{(n)}, \quad (Cu)^{(n)} + Cu^{(n)}$$

Let us derive a formula (called the *Leibniz rule*, or *Leibniz formula*) that will enable us to calculate the nth derivative of the product of two functions $u(x)$ $v(x)$. To obtain this formula, let us first find several derivatives and then establish the general rule for finding the derivative of any order:

$$\begin{aligned}
y &= uv \\
y' &= u'v + uv' \\
y'' &= u''v + u'v' + u'v' + uv'' = u''v + 2u'v' + uv'' \\
y''' &= u'''v + u''v' + 2u''v' + 2u'v'' + u'v'' + uv''' \\
&= u'''v + 3u''v' + 3u'v'' + uv''' \\
y^{\mathrm{IV}} &= u^{\mathrm{IV}}v + 4u'''v' + 6u''v'' + 4u'v''' + uv^{\mathrm{IV}}
\end{aligned}$$

The rule for forming derivatives holds for the derivative of any order and obviously consists in the following.

The expression $(u+v)^n$ is expanded by the binomial theorem, and in the expansion obtained the exponents of the powers of u and v are replaced by indices that are the orders of the derivatives, and the zero powers $(u^0=v^0=1)$ in the end terms of the expansion are replaced by the functions themselves (that is, "derivatives of zero order"):

$$y^{(n)}=(uv)^{(n)}=u^{(n)}v+nu^{(n-1)}v'+\frac{n(n-1)}{1\cdot 2}u^{(n-2)}v''+\ldots+uv^{(n)}$$

This is the *Leibniz rule*.

A rigorous proof of this formula may be carried out by the method of complete mathematical induction (in other words, to prove that if this formula holds for the nth order it will hold for the order $n+1$).

Example 3. $y=e^{ax}x^2$. Find the derivative $y^{(n)}$.

Solution.

$$\begin{array}{ll} u=e^{ax}, & v=x^2 \\ u'=ae^{ax}, & v'=2x \\ u''=a^2e^{ax}, & v''=2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ u^{(n)}=a^ne^{ax}, & v'''=v^{\mathrm{IV}}=\ldots=0 \end{array}$$

$$y^{(n)}=a^ne^{ax}x^2+na^{n-1}e^{ax}\cdot 2x+\frac{n(n-1)}{1\cdot 2}a^{n-2}e^{ax}\cdot 2$$

or

$$y^{(n)}=e^{ax}\left[a^nx^2+2na^{n-1}x+n(n-1)a^{n-2}\right]$$

3.23 DIFFERENTIALS OF DIFFERENT ORDERS

Suppose we have a function $y=f(x)$, where x is the independent variable. The differential of this function

$$dy=f'(x)\,dx$$

is some function of x, but only the first factor, $f'(x)$, can depend on x; the second factor, dx, is an increment of the independent variable x and is independent of the value of this variable. Since dy is a function of x we have the right to speak of the differential of this function.

The differential of the differential of a function is called the *second differential* or the *second-order differential* of the function and is denoted by d^2y:

$$d(dy)=d^2y$$

Let us find the expression for the second differential. By virtue of the general definition of a differential we have

$$d^2y = [f'(x)\,dx]'\,dx$$

Since dx is independent of x, dx is taken outside the sign of the derivative upon differentiation, and we get

$$d^2y = f''(x)\,(dx)^2$$

When writing the degree of a differential it is common to drop the brackets; in place of $(dx)^2$ we write dx^2 to mean the square of the expression dx; in place of $(dx)^3$ we write dx^3, etc.

The *third differential* or the *third-order differential* of a function is the differential of its second differential:

$$d^3y = d\,(d^2y) = [f''(x)\,dx^2]'\,dx = f'''(x)\,dx^3$$

Generally, the *nth differential* is the first differential of a differential of the $(n-1)$th order:

$$d^ny = d\,(d^{n-1}y) = [f^{(n-1)}(x)\,dx^{n-1}]'\,dx$$

$$d^ny = f^{(n)}(x)\,dx^n \tag{1}$$

Using differentials of different orders, we can represent the derivative of any order as a ratio of differentials of the appropriate orders:

$$f'(x) = \frac{dy}{dx}, \quad f''(x) = \frac{d^2y}{dx^2}, \quad \ldots, \quad f^{(n)}(x) = \frac{d^ny}{dx^n} \tag{2}$$

Note. Equations (1) and (2) are true (for $n > 1$) solely in the case where x is an independent variable. Indeed, suppose we have the composite function

$$y = F(u), \quad u = \varphi(x) \tag{3}$$

We have seen that the first-order differential preserves its form irrespective of whether u is an independent variable or a function of x:

$$dy = F'_u(u)\,du \tag{4}$$

The second and higher differentials do not have that property. Indeed, by (3) and (4), we get

$$d^2y = d\,(F'_u(u)\,du)$$

But here $du = \varphi'(x)\,dx$ is dependent on x and so we get

$$d^2y = d\,(F'_u(u))\,du + F'_u(u)\,d\,(du)$$

or

$$d^2y = F''_{uu}(u)\,(du)^2 + F'_u(u)\,d^2u, \text{ where } d^2u = \varphi''(x)\,(dx)^2 \tag{5}$$

In similar fashion, we find d^3y and so on.

Example 1. Find dy and d^2y of the composite function

$$y=\sin u, \ u=\sqrt{x}$$

Solution.

$$dy=\cos u\cdot\frac{1}{2\sqrt{x}}\quad dx=\cos u\,du$$

Furthermore, by formula (5), we obtain

$$d^2y=-\sin u\,(du)^2+\cos u\,d^2u=-\sin u\,(du)^2+\cos u\cdot u''\,(dx)^2$$
$$=-\sin u\left(\frac{1}{2\sqrt{x}}\right)^2(dx)^2+\cos u\left(-\frac{1}{4x^{3/2}}\right)(dx)^2$$

DERIVATIVES (OF VARIOUS ORDERS) OF IMPLICIT FUNCTIONS AND OF FUNCTIONS REPRESENTED PARAMETRICALLY

1. An example will illustrate the finding of derivatives of different orders of *implicit functions.*

Let an implicit function y of x be defined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \tag{1}$$

Differentiate all terms of the equation, with respect to x, and remember that y is a function of x:

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

From this we get

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y} \tag{2}$$

Again differentiate with respect to x (having in view that y is a function of x):

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\cdot\frac{y-x\dfrac{dy}{dx}}{y^2}$$

Substituting, in place of the derivative $\frac{dy}{dx}$, its expression from (2), we get

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\cdot\frac{y+x\dfrac{b^2}{a^2}\dfrac{x}{y}}{y^2}$$

or, after simplifying.

$$\frac{d^2y}{dx^2}=-\frac{b^2(a^2y^2+b^2x^2)}{a^4y^3}\,.$$

From equation (1) it follows that

$$a^2y^2+b^2x^2=a^2b^2$$

Therefore the second derivative may be represented as

$$\frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}.$$

Differentiating the latter equation with respect to x, we find $\frac{d^3y}{dx^3}$, etc.

2. Let us now consider the problem of finding the derivatives of higher orders **of a function represented parametrically.**

Let the function y of x be represented by parametric equations:

$$\left.\begin{array}{l} x=\varphi(t) \\ y=\psi(t) \end{array}\right\} t_0\leqslant t\leqslant T \tag{3}$$

The function $x=\varphi(t)$ has an inverse function $t=\Phi(x)$ on the interval $[t_0, T]$.

In Sec 3.18 it was proved that in this case the derivative $\frac{dy}{dx}$ is defined by the equation

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{4}$$

To find the second derivative, $\frac{d^2y}{dx^2}$, differentiate (4) with respect to x, bearing in mind that t is a function of x:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)\frac{dt}{dx} \tag{5}$$

but

$$\frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)=\frac{\frac{dx}{dt}\frac{d}{dt}\left(\frac{dy}{dt}\right)-\frac{dy}{dt}\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}$$

$$\frac{dt}{dx}=\frac{1}{\frac{dx}{dt}}$$

Substituting the latter expressions into (5), we get

$$\frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

This formula may be written in more compact form as follows:

$$\frac{d^2y}{dx^2}=\frac{\varphi'(t)\,\psi''(t)-\psi'(t)\,\varphi''(t)}{[\varphi'(t)]^3}$$

In similar fashion we can find the derivatives $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ and so forth.

Example. A function y of x is represented parametrically:

$$x=a\cos t,\quad y=b\sin t$$

Find the derivatives $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.

Solution.

$$\frac{dx}{dt}=-a\sin t,\quad \frac{d^2x}{dt^2}=-a\cos t$$

$$\frac{dy}{dt}=b\cos t,\quad \frac{d^2y}{dt^2}=-b\sin t$$

$$\frac{dy}{dx}=\frac{b\cos t}{-a\sin t}=-\frac{b}{a}\cot t$$

$$\frac{d^2y}{dx^2}=\frac{(-a\sin t)(-b\sin t)-(b\cos t)(-a\cos t)}{(-a\sin t)^3}=-\frac{b}{a^2}\frac{1}{\sin^3 t}$$

THE MECHANICAL MEANING OF THE SECOND DERIVATIVE

Let s be the path covered by a body in translation as a function of the time; it is expressed as

$$s=f(t) \tag{1}$$

As we already know (see Sec. 3.1.), the velocity v of a body at any time is equal to the first derivative of the path with respect to time:

$$v=\frac{ds}{dt} \tag{2}$$

At some time t, let the velocity of the body be v. If the motion is not uniform, then during an interval of time Δt that has elapsed since t the velocity will change by the increment Δv.

The *average acceleration* during time Δt is the ratio of the increment in velocity Δv to the increment in time:

$$a_{av}=\frac{\Delta v}{\Delta t}$$

Acceleration at a given instant is the limit of the ratio of the increment in velocity to the increment in time as the latter approaches zero:

$$a=\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}$$

In other words, acceleration (at a given instant) is equal to the derivative of the velocity with respect to time:

$$a = \frac{dv}{dt}$$

but since $v = \frac{ds}{dt}$, consequently,

$$a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

or *the acceleration of linear motion is equal to the second derivative of the path covered with respect to time.* Reverting to equation (1), we get

$$a = f''(t)$$

Example. Find the velocity v and the acceleration a of a freely falling body, if the dependence of distance s upon time t is given by the formula

$$s = \frac{1}{2} gt^2 + v_0 t + s_0 \qquad (3)$$

where $g = 9.8$ m/sec^2 is the acceleration of gravity and $s_0 = s_{t=0}$ is the value of s at $t = 0$.

Solution. Differentiating, we find

$$v = \frac{ds}{dt} = gt + v_0 \qquad (4)$$

from this formula it follows that $v_0 = (v)_{t=0}$.

Differentiating again, we find

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = g$$

Let it be noted that, conversely, if the acceleration of some motion is constant and equal to g, the velocity will be expressed by equation (4), and the distance by equation (3) provided that $(v)_{t=0} = v_0$ and $(s)_{t=0} = s_0$.

THE EQUATIONS OF A TANGENT AND OF A NORMAL. THE LENGTHS OF A SUBTANGENT AND A SUBNORMAL

Let us consider a curve whose equation is

$$y = f(x)$$

On this curve take a point $M(x_1, y_1)$ (Fig. 88) and write the equation of the tangent line to the given curve at the point M, assuming that this tangent is not parallel to the axis of ordinates.

The equation of a straight line with slope k passing through the point M is of the form

$$y - y_1 = k(x - x_1)$$

For the tangent line

$$k = f'(x_1)$$

and so the **equation of the tangent** is of the form

$$y - y_1 = f'(x_1)(x - x_1)$$

In addition to the tangent to a curve at a given point, one often has to consider the normal.

Definition. The *normal* to a curve at a given point is a straight line passing through the given point perpendicular to the tangent at that point.

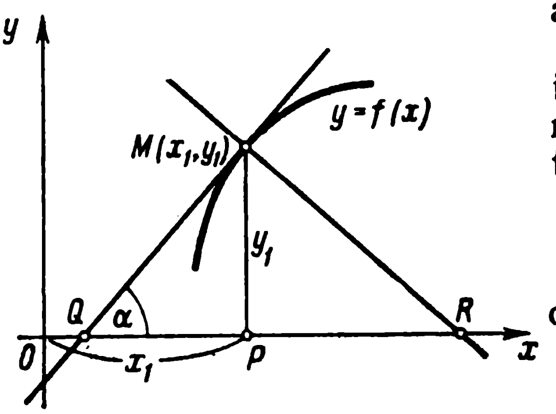

Fig. 88

From the definition of a normal it follows that its slope k_n is connected with the slope k_t of the tangent by the equation

$$k_n = -\frac{1}{k_t}$$

or

$$k_n = -\frac{1}{f'(x_1)}$$

Hence, the **equation of a normal** to a curve $y = f(x)$ at a point $M(x_1, y_1)$ is of the form

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)$$

Example 1. Write the equations of the tangent and the normal to the curve $y = x^3$ at the point $M(1, 1)$.

Solution. Since $y' = 3x^2$, the slope of the tangent is $(y')_{x=1} = 3$.

Therefore, the equation of the tangent is

$$y - 1 = 3(x - 1) \text{ or } y = 3x - 2$$

The equation of the normal is

$$y - 1 = -\frac{1}{3}(x - 1)$$

or

$$y = -\frac{1}{3}x + \frac{4}{3}$$

(see Fig. 89).

The length T of the segment QM (Fig. 88) of the tangent between the point of tangency and the x-axis is called the *length of the tangent*. The projection of this segment on the x-axis, that is,

QP, is called the *subtangent*; the length of the subtangent is denoted by S_T. The length N of the segment MR is called the *length of the normal*, while the projection RP of the segment RM on the x-axis is called the *subnormal*; the length of the subnormal is denoted by S_N.

Let us find the quantities T, S_T, N, S_N for the curve $y=f(x)$ and the point $M(x_1, y_1)$.

From Fig. 88 it will be seen that

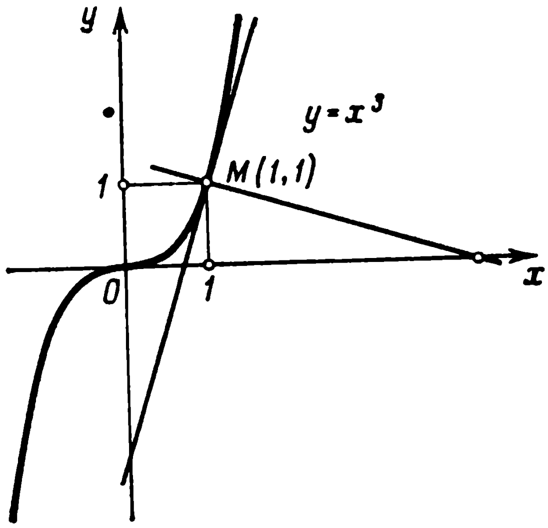

Fig. 89

$$QP=|y_1 \cot\alpha|=\left|\frac{y_1}{\tan\alpha}\right|=\left|\frac{y_1}{y_1'}\right|$$

therefore

$$S_T=\left|\frac{y_1}{y_1'}\right|$$

$$T=\sqrt{y_1^2+\frac{y_1^2}{y_1'^2}}=\left|\frac{y_1}{y_1'}\sqrt{y_1'^2+1}\right|$$

It is further clear from this same figure that

$$PR=|y_1 \tan\alpha|=|y_1y_1'|$$

and so

$$S_N=|y_1y_1'|$$

$$N=\sqrt{y_1^2+(y_1y_1')^2}=\left|y_1\sqrt{1+y_1'^2}\right|$$

These formulas are derived on the assumption that $y_1>0$, $y_1'>0$. However, they hold in the general case as well.

Example 2. Find the equations of the tangent and normal, the lengths of the tangent and the subtangent, the lengths of the normal and subnormal for

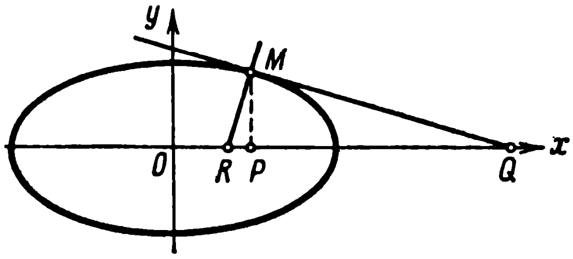

Fig. 90

the ellipse

$$x=a\cos t, \quad y=b\sin t \tag{1}$$

at the point $M(x_1, y_1)$ for which $t=\frac{\pi}{4}$ (Fig. 90).

Solution. From equations (1) we find

$$\frac{dx}{dt}=-a\sin t,\quad \frac{dy}{dt}=b\cos t,\quad \frac{dy}{dx}=-\frac{b}{a}\cot t,\quad \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}}=-\frac{b}{a}$$

We find the coordinates of the point of tangency M:

$$x_1=(x)_{t=\frac{\pi}{4}}=\frac{a}{\sqrt{2}},\quad y_1=(y)_{t=\frac{\pi}{4}}=\frac{b}{\sqrt{2}}$$

The equation of the tangent is

$$y-\frac{b}{\sqrt{2}}=-\frac{b}{a}\left(x-\frac{a}{\sqrt{2}}\right)$$

or

$$bx+ay-ab\sqrt{2}=0$$

The equation of the normal is

$$y-\frac{b}{\sqrt{2}}=\frac{a}{b}\left(x-\frac{a}{\sqrt{2}}\right)$$

or

$$(ax-by)\sqrt{2}-a^2+b^2=0$$

The lengths of the subtangent and subnormal are

$$S_T=\left|\frac{\frac{b}{\sqrt{2}}}{-\frac{b}{a}}\right|=\frac{a}{\sqrt{2}}$$

$$S_N=\left|\frac{b}{\sqrt{2}}\left(-\frac{b}{a}\right)\right|=\frac{b^2}{a\sqrt{2}}$$

The lengths of the tangent and the normal are

$$T=\left|\frac{\frac{b}{\sqrt{2}}}{-\frac{b}{a}}\sqrt{\left(-\frac{b}{a}\right)^2+1}\right|=\frac{1}{\sqrt{2}}\sqrt{a^2+b^2}$$

$$N=\left|\frac{b}{\sqrt{2}}\sqrt{1+\left(-\frac{b}{a}\right)^2}\right|=\frac{b}{a\sqrt{2}}\sqrt{a^2+b^2}$$

THE GEOMETRIC MEANING OF THE DERIVATIVE OF THE RADIUS VECTOR WITH RESPECT TO THE POLAR ANGLE

We have the following equation of a curve in polar coordinates:

$$\rho=f(\theta) \tag{1}$$

Let us write the formulas for changing from polar coordinates to rectangular Cartesian coordinates:

$$x=\rho\cos\theta,\quad y=\rho\sin\theta$$

Substituting, in place of ρ, its expression in terms of θ from equation (1), we get

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta \tag{2}$$

Equations (2) are parametric equations of the given curve, the parameter being the polar angle θ (Fig. 91).

If we denote by φ the angle formed by the tangent to the curve at some point $M(\rho, \theta)$ with the positive x-axis, we will have

$$\tan\varphi = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

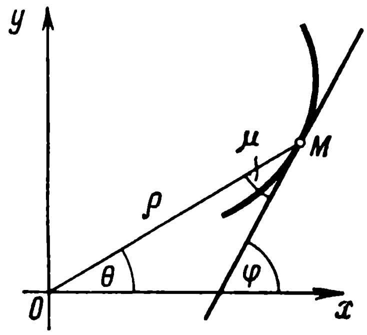

Fig. 91

or

$$\tan\varphi = \frac{\frac{d\rho}{d\theta}\sin\theta + \rho\cos\theta}{\frac{d\rho}{d\theta}\cos\theta - \rho\sin\theta} \tag{3}$$

Denote by μ the angle between the direction of the radius vector and the tangent. It is obvious that $\mu = \varphi - \theta$,

$$\tan\mu = \frac{\tan\varphi - \tan\theta}{1 + \tan\varphi\tan\theta}$$

Substituting, in place of $\tan\varphi$, its expression (3) and making the necessary changes, we get

$$\tan\mu = \frac{(\rho'\sin\theta + \rho\cos\theta)\cos\theta - (\rho'\cos\theta - \rho\sin\theta)\sin\theta}{(\rho'\cos\theta - \rho\sin\theta)\cos\theta + (\rho'\sin\theta + \rho\cos\theta)\sin\theta} = \frac{\rho}{\rho'}$$

or

$$\rho'_\theta = \rho\cot\mu \tag{4}$$

Thus, the derivative of the radius vector with respect to the polar angle is equal to the length of the radius vector multiplied by the cotangent of the angle between the radius vector and the tangent to the curve at the given point.

Example. Show that the tangent to the logarithmic spiral

$$\rho = e^{a\theta}$$

intersects the radius vector at a constant angle.

Solution. From the equation of the spiral we get

$$\rho' = ae^{a\theta}$$

From formula (4) we have

$$\cot\mu = \frac{\rho'}{\rho} = a, \text{ that is, } \mu = \operatorname{arccot} a = \text{const}$$