Lecture 1

NUMBER. VARIABLE. FUNCTION

REAL NUMBERS. REAL NUMBERS AS POINTS ON A NUMBER SCALE

Number is one of the basic concepts of mathematics. It originated in ancient times and has undergone expansion and generalization over the centuries.

Whole numbers and fractions, both positive and negative, together with the number zero are called *rational numbers*. Every rational number may be represented in the form of a ratio, $\frac{p}{q}$, of two integers p and q; for example,

$$\frac{5}{7}, \quad 1.25=\frac{5}{4}$$

In particular, the integer p may be regarded as a ratio of two integers $\frac{p}{1}$; for example,

$$6=\frac{6}{1}, \quad 0=\frac{0}{1}$$

Rational numbers may be represented in the form of periodic terminating or nonterminating fractions. Numbers represented by nonterminating, but nonperiodic, decimal fractions are called *irrational numbers*; such are the numbers $\sqrt{2}$, $\sqrt{3}$, $5-\sqrt{2}$, etc.

The collection of all rational and irrational numbers makes up the set of *real numbers*. The real numbers are *ordered in magnitude*; that is to say, for each pair of real numbers x and y there is one, and only one, of the following relations:

$$x<y, \quad x=y, \quad x>y$$

Real numbers may be depicted as points on a number scale. A *number scale* is an infinite straight line on which are chosen: (1) a certain point O called the origin, (2) a positive direction indicated by an arrow, and (3) a suitable unit of length. We shall usually make the number scale horizontal and take the positive direction to be from left to right.

If the number x_1 is positive, it is depicted as a point M_1 at a distance $OM_1=x_1$ to the right of the origin O; if the number x_2

is negative, it is represented by a point M_2 to the left of O at a distance $OM_2 = -x_2$ (Fig. 1). The point O represents the number zero. It is obvious that every real number is represented by a definite point on the number scale. Two different real numbers are represented by different points on the number scale.

The following assertion is also true: each point on the number scale represents only one real number (rational or irrational).

To summarize, all real numbers and all points on the number scale are in one-to-one correspondence: to each number there corresponds only one point, and conversely, to each point there corresponds only one number. This frequently enables us to regard "the number x" and "the point x" as, in a certain sense, equivalent expressions. We shall make wide use of this circumstance in our course.

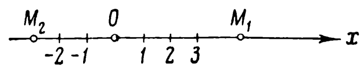

Fig. 1

We state without proof the following important property of the set of real numbers: *both rational and irrational numbers may be found between any two arbitrary real numbers.* In geometrical terms, this proposition reads thus: *both rational and irrational points may be found between any two arbitrary points on the number scale.*

In conclusion we give the following theorem, which, in a certain sense, represents a bridge between theory and practice.

Theorem. *Every irrational number α may be expressed, to any degree of accuracy, with the aid of rational numbers.*

Indeed, let the irrational number $\alpha > 0$ and let it be required to evaluate α with an accuracy of $\frac{1}{n}$ $\left(\text{for example, } \frac{1}{10}, \frac{1}{100}, \text{ and so forth}\right)$.

No matter what α is, it lies between two integral numbers N and $N+1$. We divide the interval between N and $N+1$ into n parts; then α will lie somewhere between the rational numbers $N+\frac{m}{n}$ and $N+\frac{m+1}{n}$. Since their difference is equal to $\frac{1}{n}$, each of them expresses α to the given degree of accuracy, the former being too small and the latter, too large.

Example. The irrational number $\sqrt{2}$ is expressed by the rational numbers:
1.4 and 1.5 to one decimal place,
1.41 and 1.42 to two decimal places,
1.414 and 1.415 to three decimal places, etc.

THE ABSOLUTE VALUE OF A REAL NUMBER

Let us introduce a concept which we shall need later on: the absolute value of a real number.

Definition. *The absolute value* (or *modulus*) of a real number x (written $|x|$) is a nonnegative real number that satisfies the conditions

$$|x| = x \quad \text{if } x \geqslant 0$$
$$|x| = -x \quad \text{if } x < 0$$

Examples. $|2| = 2$, $|-5| = 5$, $|0| = 0$.

From the definition it follows that the relationship $x \leqslant |x|$ holds for any x.

Let us examine some of the properties of absolute values.

1. *The absolute value of an algebraic sum of several real numbers is no greater than the sum of the absolute values of the terms*

$$|x+y| \leqslant |x| + |y|$$

Proof. Let $x+y \geqslant 0$, then

$$|x+y| = x+y \leqslant |x| + |y| \text{ (since } x \leqslant |x| \text{ and } y \leqslant |y|)$$

Let $x+y < 0$, then

$$|x+y| = -(x+y) = (-x) + (-y) \leqslant |x| + |y|$$

This completes the proof.

The foregoing proof is readily extended to any number of terms.

Examples.

$$|-2+3| < |-2| + |3| = 2+3 = 5 \text{ or } 1 < 5,$$
$$|-3-5| = |-3| + |-5| = 3+5 = 8 \text{ or } 8 = 8.$$

2. *The absolute value of a difference is no less than the difference of the absolute values of the minuend and subtrahend:*

$$|x-y| \geqslant |x| - |y|, \quad |x| > |y|$$

Proof. Let $x - y = z$, then $x = y + z$ and from what has been proved

$$|x| = |y+z| \leqslant |y| + |z| = |y| + |x-y|$$

whence

$$|x| - |y| \leqslant |x-y|$$

thus completing the proof.

3. *The absolute value of a product is equal to the product of the absolute values of the factors:*

$$|xyz| = |x|\ |y|\ |z|$$

4. *The absolute value of a quotient is equal to the quotient of the absolute values of the dividend and the divisor:*

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

The latter two properties follow directly from the definition of absolute value.

VARIABLES AND CONSTANTS

The numerical values of such physical quantities as time, length, area, volume, mass, velocity, pressure, temperature, etc. are determined by measurement. Mathematics deals with quantities divested of any specific content. From now on, when speaking of quantities, we shall have in view their numerical values. In various phenomena, the numerical values of certain quantities vary, while the numerical values of others remain fixed. For instance, in the uniform motion of a point, time and distance change, while the velocity remains constant.

A *variable* is a quantity that takes on various numerical values. A *constant* is a quantity whose numerical values remain fixed. We shall use the letters x, y, z, u, $\ldots$, etc. to designate variables, and the letters a, b, c, $\ldots$, etc. to designate constants.

Note. In mathematics, a constant is frequently regarded as a special case of variable whose numerical values are the same.

It should be noted that when considering specific physical phenomena it may happen that one and the same quantity in one phenomenon is a constant while in another it is a variable. For example, the velocity of uniform motion is a constant, while the velocity of uniformly accelerated motion is a variable. Quantities that have the same value under all circumstances are called *absolute constants*. For example, the ratio of the circumference of a circle to its diameter is an absolute constant: $\pi = 3.14159\ldots$

As we shall see throughout this course, the concept of a variable quantity is the basic concept of differential and integral calculus. In "Dialectics of Nature", Friedrich Engels wrote: "The turning point in mathematics was Descartes' variable magnitude. With that came *motion* and hence *dialectics* in mathematics, and *at once*, too, *of necessity* the differential and integral calculus."

THE RANGE OF A VARIABLE

A variable takes on a series of numerical values. The collection of these values may differ depending on the character of the problem. For example, the temperature of water heated under ordinary conditions will vary from room temperature (15-18 °C) to the boiling point, 100 °C. The variable quantity $x = \cos\alpha$ can take on all values from -1 to $+1$.

The values of a variable are geometrically depicted as points on a number scale. For instance, the values of the variable $x = \cos\alpha$ for all possible values of α are depicted as the set of points of the interval from -1 to 1, including the points -1 and 1 (Fig. 2).

Definition. The set of all numerical values of a variable quantity is called the *range* of the variable.

We shall now define the following ranges of a variable that will be frequently used later on.

An *interval* is the set of all numbers x lying between the given points a and b *(the end points)* and is called closed or open accordingly as it does or does not include its end points.

An *open interval* is the collection of all numbers x lying between and *excluding* the given numbers a and b $(a<b)$; it is denoted (a, b) or by means of the inequalities $a<x<b$.

A *closed interval* is the set of all numbers x lying between and *including* the two given numbers a and b; it is denoted $[a, b]$ or, by means of inequalities, $a \leqslant x \leqslant b$.

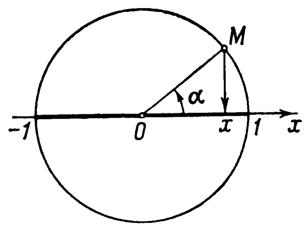

Fig. 2

If one of the numbers a or b (say, a) belongs to the interval, while the other does not, we have a *partly closed (half-closed) interval*, which may be given by the inequalities $a \leqslant x<b$ and is denoted $[a, b)$. If the number b belongs to the set and a does not, we have the half-closed interval $(a, b]$, which may be given by the inequalities $a<x \leqslant b$.

If the variable x assumes all possible values greater than a, such an interval is denoted $(a, +\infty)$ and is represented by the conditional inequalities $a<x<+\infty$. In the same way we regard the infinite intervals and half-closed infinite intervals represented by the conditional inequalities

$$a \leqslant x<+\infty, \quad -\infty<x<c, \quad -\infty<x \leqslant c, \quad -\infty<x<+\infty$$

Example. The range of the variable $x=\cos \alpha$ for all possible values of α is the interval $[-1, 1]$ and is defined by the inequalities $-1 \leqslant x \leqslant 1$.

The foregoing definitions may be formulated for a "point" in place of a "number".

The *neighbourhood* of a given point x_0 is an arbitrary interval $(a\ b)$ containing this point within it; that is, the interval (a, b) whose end points satisfy the condition $a<x_0<b$. One often considers the neighbourhood (a, b) of the point x_0 for which x_0 is the midpoint. Then x_0 is called the *centre of the neighbourhood* and the quantity $\frac{b-a}{2}$, the *radius of the neighbourhood*. Fig. 3 shows the neighbourhood $(x_0-\varepsilon, x_0+\varepsilon)$ of the point x_0 with radius ε.

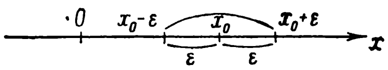

Fig. 3

ORDERED VARIABLES. INCREASING AND DECREASING VARIABLES. BOUNDED VARIABLES

We shall say that the variable x is an *ordered variable quantity* if its range is known and if about each of any two of its values it may be said which value is the preceding one and which is the following one. Here, the notions "preceding" and "following" are not connected with time but serve as a way to "order" the values of the variable, i. e., to establish the order of the respective values of the variable.

A particular case of an ordered variable is a variable whose values form a *number sequence* $x_1, x_2, x_3, \ldots, x_n, \ldots$. Here, for $k' < k$, the value $x_{k'}$ is the preceding value, and the value x_k is the following value, irrespective of which one is the greater.

Definition 1. A variable is called *increasing* if each subsequent value of it is greater than the preceding value. A variable is called *decreasing* if each subsequent value is less than the preceding value.

Increasing variable quantities and decreasing variable quantities are called *monotonically varying* variables or simply *monotonic quantities*.

Example. When the number of sides of a regular polygon inscribed in a circle is doubled, the area s of the polygon is an increasing variable. The area of a regular polygon circumscribed about a circle, when the number of sides is doubled, is a decreasing variable. It may be noted that not every variable quantity is necessarily increasing or decreasing. Thus, if α is an increasing variable over the interval $[0, 2\pi]$, the variable $x = \sin\alpha$ is not a monotonic quantity. It first increases from 0 to 1, then decreases from 1 to -1, and then increases from -1 to 0.

Definition 2. The variable x is called *bounded* if there exists a constant $M > 0$ such that all subsequent values of the variable, after a certain one, satisfy the condition

$$-M \leqslant x \leqslant M \quad \text{or} \quad |x| \leqslant M$$

In other words, a variable is called bounded if it is possible to indicate an interval $[-M, M]$ such that all subsequent values of the variable, after a certain one, will belong to this interval. However, one should not think that the variable will necessarily assume all values on the interval $[-M, M]$. For example, the variable that assumes all possible rational values on the interval $[-2, 2]$ is bounded, and nevertheless it does not assume all values on $[-2, 2]$, namely, it does not take on the irrational values.

FUNCTION

In the study of natural phenomena and the solution of technical and mathematical problems, one finds it necessary to consider the variation of one quantity as dependent on the variation of another.

For instance, in studies of motion, the path traversed is regarded as a variable which varies with time. Here, the path traversed is a **function** of the time.

Let us consider another example. We know that the area of a circle, in terms of the radius, is $Q=\pi R^2$. If the radius R takes on a variety of numerical values, the area Q will also assume various numerical values. Thus, the variation of one variable brings about a variation in the other. Here, the area of a circle Q is a function of the radius R. Let us formulate a definition of the concept "function".

Definition 1. If to each value of a variable x (within a certain range) there corresponds one definite value of another variable y, then y is a *function* of x or, in functional notation, $y=f(x)$, $y=\varphi(x)$, and so forth.

The variable x is called the *independent variable* or *argument*. The relation between the variables x and y is called a *functional relation*. The letter f in the functional notation $y=f(x)$ indicates that some kind of operations must be performed on the value of x in order to obtain the value of y. In place of the notation $y=f(x)$, $u=\varphi(x)$, etc. one occasionally finds $y=y(x)$, $u=u(x)$, etc. the letters y, u designating both the dependent variable and the symbol of the operations to be performed on x.

The notation $y=C$, where C is a constant, denotes a function whose value for any value of x is the same and is equal to C.

Definition 2. The set of values of x for which the values of the function y are determined by the rule $f(x)$ is called the *domain of definition of the function*.

Example 1. The function $y=\sin x$ is defined for all values of x. Therefore, its domain of definition is the infinite interval $-\infty < x < +\infty$.

Note 1. If we have a function relation of two variable quantities x and $y=f(x)$ and it x and $y=f(x)$ are regarded as ordered variables, then of the two values of the function $y^*=f(x^*)$ and $y^{**}=f(x^{**})$ corresponding to two values of the argument x^* and x^{**}, the subsequent value of the function will be that one which corresponds to the subsequent value of the argument. The following definition is therefore natural.

Definition 3. If the function $y=f(x)$ is such that to a greater value of the argument x there corresponds a greater value of the function, then the function $y=f(x)$ is called *increasing*. A *decreasing* function is similarly defined.

Example 2. The function $Q=\pi R^2$ for $0 < R < \infty$ is an increasing function because to a greater value of R there corresponds a greater value of Q.

Note 2. The definition of a function is sometimes broadened so that to each value of x, within a certain range, there corresponds not one but several values of y or even an infinitude of values of y. In this case we have a *multiple-valued* function in contrast to the one defined above, which is called a *single-valued* function. Henceforward, when speaking of a function, we shall have in view only **single-valued** functions. If it becomes necessary to deal with multiple-valued functions we shall specify this fact.

WAYS OF REPRESENTING FUNCTIONS

I. Tabular representation of a function

Here, the values of the argument $x_1, x_2, \ldots, x_n$ and the corresponding values of the function $y_1, y_2, \ldots, y_n$ are written out in a definite order.

x	x_1	x_2	....	x_n
y	y_1	y_2	....	y_n

Examples are tables of trigonometric functions, tables of logarithms, and so on.

An experimental study of phenomena can result in tables that express a functional relation between the measured quantities. For example, temperature measurements of the air at a meteorological station on a definite day yield a table like the following.

The temperature T (in degrees) is dependent on the time t (in hours).

t	1	2	3	4	5	6	7	8	9
T	0	−1	−2	−2	−0.5	1	3	3.5	4

This table defines T as a function of t.

II. Graphical representation of a function

If in a rectangular coordinate system on a plane we have a set of points $M(x, y)$ and no two points lie on a straight line parallel to the y-axis, this set of points defines a certain single-valued function $y=f(x)$; the abscissas of the points are the values of the argument, the corresponding ordinates are the values of the function (Fig. 4).

The collection of points in the xy-plane whose abscissas are the values of the independent variable and whose ordinates are the corresponding values of the function is called a *graph* of the given function.

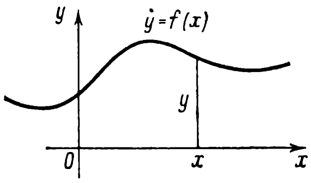

Fig. 4

III. Analytical representation of a function

Let us first explain what "analytical expression" means. By *analytical expression* we will understand a series of symbols denoting certain mathematical operations that are performed in a definite sequence on numbers and letters which designate constant or variable quantities.

By totality of known mathematical operations we mean not only the mathematical operations familiar from the course of secondary school (addition, subtraction, extraction of roots, etc.) but also those which will be defined as we proceed in this course.

The following are examples of analytical expressions:

$$x^4-2, \quad \frac{\log x-\sin x}{5x^2+1}, \quad 2^x-\sqrt{5+3x}$$

If the functional relation $y=f(x)$ is such that f denotes an analytical expression, we say that the function y of x is *represented* or *defined analytically*.

Examples of functions defined analytically are: (1) $y=x^4-2$, (2) $y=\frac{x+1}{x-1}$, (3) $y=\sqrt{1-x^2}$, (4) $y=\sin x$, (5) $Q=\pi R^2$, and so forth.

Here, the functions are defined analytically by means of a single formula (a formula is understood to be an equality of two analytical expressions). In such cases one may speak of the *natural* domain of definition of the function.

The set of values of x for which the analytical expression on the right-hand side has a definite value is the *natural domain of definition of a function* represented analytically. Thus, the natural domain of definition of the function $y=x^4-2$ is the infinite interval $-\infty<x<+\infty$, because the function is defined for all values of x. The function $y=\frac{x+1}{x-1}$ is defined for all values of x, with the exception of $x=1$, because for this value of x the denominator vanishes. For the function $y=\sqrt{1-x^2}$, the natural domain of definition is the closed interval $-1\leqslant x\leqslant 1$, and so on.

Note. It is sometimes necessary to consider only a part of the natural domain of a function, and not the whole domain. For

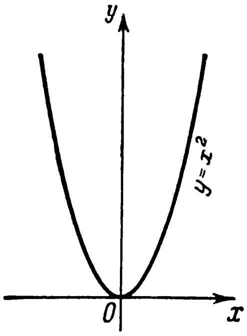

Fig. 5

instance, the dependence of the area Q of a circle upon the radius R is defined by the function $Q=\pi R^2$. The domain of this function, when considering the given geometrical problem, is the infinite interval $0 < R < +\infty$. But the natural domain of this function is the infinite interval $-\infty < R < +\infty$.

If the function $y=f(x)$ is represented analytically, it may be shown graphically on a coordinate xy-plane. Thus, the graph of the function $y=x^2$ is a parabola as shown in Fig. 5.